மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc. MATHEMATICS

I YEAR

INTEGRAL CALCULUS

Sub. Code: JMMA22

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B.Sc. MATHEMATICS –I YEAR

JMMA22: INTEGRAL CALCULUS

SYLLABUS

UNIT-I

Reduction formulae -Types, integration of product of powers of algebraic and trigonometric functions, integration of product of powers of algebraic and logarithmic functions - Bernoulli's formula.

Chapter 1: Sections 1.1- 1.4

UNIT-II

Multiple Integrals - definition of double integrals - evaluation of double integrals – double integrals in polar coordinates - Change of order of integration.

Chapter 2: Sections 2.1 - 2.5

UNIT-III

Triple integrals – applications of multiple integrals - volumes of solids of revolution - areas of curved surfaces – change of variables - Jacobian.

Chapter 3: Sections 3.1 - 3.5

UNIT-IV

Beta and Gamma functions – infinite integral – definitions – recurrence formula of Gamma functions – properties of Beta and Gamma functions - relation between Beta and Gamma functions - Applications.

Chapter 4: Sections 4.1 – 4.6

UNIT-V

Geometric and Physical Applications of Integral calculus.

Chapter 5: Sections 5.1 -5.4 , Chapter 6: Sections 6.1- 6.5

Text Book

1. Narayanan S and Manicavachagom Pillay T.K. Calculus-Volume II, (2009), S.Viswananthan Printers Pvt. Ltd.

JMMA22: INTEGRAL CALCULUS

CONTENTS

UNIT-I

Reduction formulae -Types, integration of product of powers of algebraic and trigonometric functions, integration of product of powers of algebraic and logarithmic functions - Bernoulli's formula.

Chapter1: Sections 1.1- 1.4

1.1.Reduction Formulae

A Reduction formula expresses an integral I_n that depends on Some integer n in terms of another integral I_m that involves a smaller integer m. If one repeatedly applied this formula, one may then express I_n in terms of a much simpler integral.

1.2.Integration of Product of Powers of Algebraic and Trigonometric Functions:

Result 1:

$I_n = \int x^n e^{ax} dx$, where n is a positive integer.

Here $dv = e^{ax} dx$,

$v = \int e^{ax} dx = \frac{e^{ax}}{a}$

$u = x^n$

$\therefore I_n = \int x^n \, d\left(\frac{e^{ax}}{a}\right) = \frac{e^{ax}}{a}. x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx = \frac{e^{ax}}{a}. x^n - \frac{n}{a} I_{n-1}.$

The auxiliary integrals of the same types as the given integral but with index n reduced by 1. Such a formula is called a Reduction formula and by successive applications, we can evaluate I_n. The ultimate integral is obviously $\int e^{ax} dx = \frac{e^{ax}}{a}$

Result 2:

$I_n = \int x^n \cos ax \, dx$, where n is a positive integer.

$I_n = \int x^n \cos ax \, dx \quad = \int x^n \, d\left(\frac{\sin ax}{a}\right)$

Here $u = x^n$, $v = \frac{\sin ax}{a}$

$= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax \, dx$

$= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} d(\frac{-\cos\ ax}{a})$

$= \frac{x^n \sin ax}{a} - \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} \int x^{n-2} \cos ax\, dx$

$= \frac{x^n \sin ax}{a} - \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} I_{n-2}$

The ultimate integral is either $\int x \cos ax\, dx$ (or) $\int \cos x\, dx$ according as n is odd or even.

(i) $\int x \cos ax\, dx = \int x\, d\left(\frac{\sin ax}{a}\right)$

$= \frac{x \sin ax}{a} - \frac{1}{a} \int \sin ax\, dx$

$= \frac{x \sin ax}{a} + \frac{1}{a^2} . \cos ax$

(ii) $\int \cos ax\, dx = \frac{\sin ax}{a}$

Result 3:

$I_n = \int \sin^n x\, dx$, where n is a positive integer.

$I_n = \int \sin^{n-1} x \sin x\, dx$

$= - \sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x \cos x) \cos x\, dx$

$= - \sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, (1- \sin^2 x)\, dx$

$= - \sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$

$= - \sin^{n-1} x \cos x + (n-1) I_{n-2} -(n-1) I_n$

$I_n = - \sin^{n-1} x \cos x + (n-1) I_{n-2} -(n-1) I_{n-2} – n I_n + I_n$

The ultimate integral is $\int \sin x\, dx$ or $\int dx$ according as n is odd (or) even.

(i.e.) -cos x (or) x

Corollary 1:

From equation, (1) $\Rightarrow I_n = \frac{-1}{n}(\sin^{n-1} x \cos x) + \frac{n-1}{n} I_{n-2}$ (by Reduction formula for $\int \sin^n x\, dx$)

$\therefore \int_0^{\pi/2} \sin^n x\, dx = \frac{-1}{n}(\sin^{n-1} x \cos x)_0^{\frac{\pi}{2}} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$

$$= 0 + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} sin^{n-2}\, x\, dx$$

$\therefore I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ (where $I_n = \int_0^{\pi/2} sin^n\, x\, dx$

Changing n to n-2, n-4, n-6, …… in successive steps

We get,

$I_{n-2} = \left(\frac{n-3}{n-2}\right) I_{n-4}$

$I_{n-4} = \left(\frac{n-5}{n-4}\right) I_{n-6}$ and so on

$\therefore I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} . I_{n-6}$

Case (i):

If n is an even positive integer, then

$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \int_0^{\frac{\pi}{2}} 1\ dx$

$\int_0^{\pi/2} sin^n\, x\, dx = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$, if n is even.

Case (ii):

If n is an odd positive integer, then

$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{4}{5} . \frac{2}{3} . \int_0^{\frac{\pi}{2}} sin\, x\ dx$

$= \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{4}{5} . \frac{2}{3} . [-\cos x]_0^{\frac{\pi}{2}}$

$\therefore \int_0^{\frac{\pi}{2}} sin\, x\ dx = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{4}{5} . \frac{2}{3} . 1$, if n is odd.

Example 1:

$\int_0^{\pi/2} sin^6 x\, dx$

Solution:

If n is even

$\int_0^{\pi/2} sin^6 x\, dx = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$

n =6 $\Rightarrow$ $\frac{6-1}{6} . \frac{6-3}{6-2} .... \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$

$\Rightarrow \int_0^{\pi/2} sin^6 x\, dx = \frac{5}{6} . \frac{3}{4} .... \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$

$= \frac{5\pi}{32}$

Example 2:

$\int_0^{\pi/2} sin^7 x\, dx$

Solution:

If n is odd

$\int_0^{\pi/2} sin^n x\, dx = \frac{n-1}{n} . \frac{n-3}{n-2} .... \frac{4}{5} . \frac{2}{3}$ (n is odd)

$\int_0^{\pi/2} sin^7 x\, dx = \frac{7-1}{7} . \frac{7-3}{7-2} .... \frac{4}{5} . \frac{2}{3}$

$= \frac{6}{7} . \frac{4}{5} .... \frac{4}{5} . \frac{2}{3}$

$= \frac{6}{7} . \frac{4}{5} . \frac{2}{3}$

$= \frac{48}{105}$

$= \frac{16}{35}$

Example 3:

If $\int sin^n x\, dx$, if n be an odd positive integer, we can directly integrate without using the reduction formulae. For instance, let us find $\int sin^5 x\, dx$.

Solution:

$\int sin^5 x\, dx = \int sin^4 x \sin x\, dx.$

$= \int (1- cos^2 x)^2 \sin x\, dx$

Let y = cos x $\Rightarrow$ dy = -sinx dx

$\Rightarrow - \int (1-y^2)^2\, dy$

$\Rightarrow -\int (1-2y^2+y^4)\, dy$

$= -y + \frac{2y^3}{3} - \frac{y^5}{5}$

$= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}$

Example 4:

Evaluate $\int_0^{\pi/2} x(1-x^2)^{1/2}\, dx$

Solution:

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

When $x = 0$, $\theta = 0$

$X = 1$, $\theta = \pi/2$

The integral becomes,

$\int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d(-\cos\theta)$

$= [\frac{-\cos^3\theta}{3}]_0^{\pi/2} = [0-(-1/3)] = \frac{1}{3}$

Result 4:

$I_n = \int \cos^n x\, dx$, where n is a positive integer.

$I_n = \int \cos^{n-1} x \cos x\, dx$

$= \int \cos^{n-1} x\, d(\sin x)$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin x (\sin x)\, dx$

$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$

$= \cos^{n-1} x \sin x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x\, dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - n I_n + I_n$

$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$ …… (1)

The ultimate integral is ∫ cos x dx (or)∫dx

(i.e.) sin x (or) x according as n is odd or even.

Corollary 2:

From equation (1)

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{\cos^{n-1} x \sin x}{n}\right)_0^{\frac{\pi}{2}} + \frac{(n-1)}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= 0 + \frac{(n-1)}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx \quad \text{as the first term vanishes at both limits}$$

$$= \frac{(n-1)}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

[∵ F is continuous function on [0,a] if $\int_0^a f(x)\, dx = \int_0^a f(a-x) dx$]

$$I_n = \int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin^n x\, dx$$

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \ldots \ldots \ldots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 & \text{if n is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \ldots \ldots \ldots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & if\ n\ is\ even \end{cases}$$

Example 5:

$$\int_0^{\pi/2} \cos^8 x\, dx$$

Solution:

$$\int_0^{\pi/2} \cos^8 x\, dx = \int_0^{\pi/2} \cos^8 \left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin^8 x\, dx$$

∵ n is even

$$\int_0^{\pi/2} sin^n x\, dx = \frac{n-1}{n}.\frac{n-3}{n-2} \dots \dots . .\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$\int_0^{\pi/2} sin^8 x\, dx = \frac{8-1}{8}.\frac{8-3}{8-2} \dots \dots . .\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$= \frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$= \frac{35\,\pi}{256}$$

Example 6:

$$\int_0^{\pi/2} cos^5 x\, dx$$

Solution:

$$\int_0^{\pi/2} cos^5 x\, dx = \int_0^{\pi/2} cos^5 \left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} sin^5 x\, dx$$

∵ n is odd

$$\int_0^{\pi/2} sin^5 x\, dx = \frac{5-1}{5}.\frac{5-3}{5-2} \dots \dots . . .\frac{4}{5}.\frac{2}{3}.1$$

$$= \frac{4}{5}.\frac{2}{3}.1$$

$$= \frac{8}{15}$$

Example 7:

In ∫ $\cos^n x$ dx, if n be an odd positive integer, we can directly integrate employing the reduction formula. For example, take ∫$\cos^7 x$ dx

Solution:

∫ $\cos^7 x$ dx = ∫ $\cos^6 x \cos x$ dx

Put y = sin x & dy = cos x dx (∵$\cos^2 x = 1 - \sin^2 x$ $(\cos^2 x)^3 = (1-\sin^2 x)^3$

= ∫ $(1-\sin^2 x)^3 \cos x$ dx

= ∫ $(1-y^2)^3$ dy

$= \int (1-y^2+3y^4-y^6)\, dy$

$= y-y^3+\frac{3y^5}{5} - \frac{y^7}{7}$

$= \sin x - \sin^3 x + \frac{3\sin^5 x}{5} - \frac{\sin^7 x}{7}$

Result 5:

$I_{m,n} = \int \sin^m x \cos^n x\, dx$ (m, n being positive integers)

Let $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$= \int \cos^{n-1} x (\sin^m x \cos x)\, dx$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int(-(n-1)\cos^{n-2} x \sin x) \frac{\sin^{m+1} x}{m+1} dx$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x \sin^2 x\, dx$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x (1-\cos^2 x)\, dx$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x\, dx - \frac{n-1}{m+1}\int \cos^n x \sin^m x\, dx$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$

$\Rightarrow (m+1) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2} - (n-1) I_{m,n}$

$\Rightarrow (m+1) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$ ………(a)

Here, the power of cos x has been reduced by 2.

We may, by a similar argument, arrive at the Reduction Formula in the Form

$(m+1) I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}$ …….(b)

Here, the power of sinx has been reduced by 2

To apply this formula, we note two cases.

Case (i): Let m (or) n be an odd integer, say n

Applying the formula (a) successively,

The ultimate integral is,

$I_{m,1}=\int \sin^m x \cos x\, dx = \frac{\sin^{m+1} x}{m+1}$

If however, m is odd, we can use (b) and the ultimate integral is

$I_{1,m} = \int \sin x \cos^n x\, dx = \frac{-\cos^{n+1} x}{n+1}$

If both m and n are odd, reduce the smaller index.

Note:

When either m (or) n (or)both are odd, we can integrate $\sin^m x \cos^n x$ directly without recourse to a reduction formula.

Case (ii): Let both m and n be even positive integers

Let n<m. Applying (a), the ultimate integral is, $I_{m,0} = \int \sin^m x\, dx$

Which has been discussed in 13.3

Similarly, $I_n = \int \sin^m x\, dx$

$\Rightarrow n\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$

Corollary 3:

$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ (m, n being positive integer)

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+1}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

$= \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$ as the first term vanishes at both limits.

$$= \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\int_0^{\pi/2} \sin^m x \cos^{n-4} x\, dx$$

$= \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\,\frac{n-5}{m+n-4}\ldots\ldots\ldots I_{m,1}$ (or) $I_{m,0}$ according as n is odd (or) even

(i) If n is odd,

$I_{m,1}= \int_0^{\pi/2} \sin^m x \cos x\, dx$

$$= [\frac{sin^{m+1}x}{m+1}]_0^{\frac{\pi}{2}} = \frac{1}{m+1}$$

When n is odd,

$$\int_0^{\pi/2} sin^m x \, cos^n \, x \, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \ldots\ldots\ldots \frac{2}{m+3} \frac{1}{m+1}$$

(ii) If n is even,

$$I_{m,0} = \int_0^{\pi/2} sin^m x \; dx = \frac{n-1}{m+n} . \frac{n-3}{m+n-2} \ldots\ldots \frac{1}{2} \frac{\pi}{2}.$$

∴ (By corollary 1)

When m is even

$$\int_0^{\pi/2} sin^m x \, cos^n \, x \, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \ldots\ldots\ldots \frac{1}{m+1} \frac{m-1}{m} \frac{m-3}{m-2} \ldots\ldots \frac{1}{2} \frac{\pi}{2}$$

Example 8:

$\int sin^6x \, cos^3x \, dx$

Solution:

Put y =sin x & dy = cos x dx

$\int sin^6x \, cos^3x \, dx = \int sin^6x \, cos^2x \, cos \, x \, dx$

$= \int sin^6x \, (1-cos^2x) \, cos \, x \, dx$

$= \int y^6 (1-y^2) \, dy$

$= \int y^6 - y^8 \, dy$

$= \frac{y^7}{7} - \frac{y^9}{9}$

$= \frac{sin^7x}{7} - \frac{sin^9x}{9}$

Example 9:

$\int sin^9x \, cos^5x \, dx$

Solution:

$\int sin^9x \, cos^5x \, dx = \int sin^9x \, cos^4x \, cos \, x \, dx$

$= \int \sin^9 x\,(1-\sin^2 x)^2 \cos x\,dx$

$= \int y^9 (1-y^2)^2\,dy$

$= \int y^9 (1-2y^2+y^4)\,dy$

$= \int (y^9-2y^{11}+y^{13})\,dy$

$$= \frac{y^{10}}{10} - \frac{2y^{12}}{12} + \frac{y^{14}}{14}$$

$$= \frac{y^{10}}{10} - \frac{y^{12}}{6} + \frac{y^{14}}{14}$$

$$= \frac{\sin^{10}x}{10} - \frac{\sin^{12}x}{6} + \frac{\sin^{14}x}{14}$$

Example 10:

$$\int_0^{\pi/2} \sin^6 x \cos^5 x\ dx$$

Solution:

Here, m=6, n=5

$$\int_0^{\pi/2} \sin^m x \cos^n x\ dx\ = \frac{n-1}{m+n}\ \frac{n-3}{m+n-2}\ldots\ldots.\frac{1}{m+1}\ \frac{m-1}{m}\frac{m-3}{m-2}\ldots..\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^6 x \cos^3 x\ dx = \frac{4}{11}.\frac{2}{9}.\frac{1}{7} = \frac{8}{693}.$$

Example 11:

$$\int_0^{\pi/2} \sin^6 x \cos^4 x\ dx$$

Solution:

Here, m=6, n=4

$$\int_0^{\pi/2} \sin^m x \cos^n x\ dx\ = \frac{n-1}{m+n}\ .\ \frac{n-3}{m+n-2}\ldots\ldots.\frac{1}{m+1}\ \frac{m-1}{m}\frac{m-3}{m-2}\ldots..\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^6 x \cos^4 x\ dx = \frac{6-1}{6+4}.\frac{6-3}{6+4-2}.3.\frac{6-3}{6-2}.\frac{1}{2}.\frac{\pi}{2}.$$

$$= \frac{5}{10}.\frac{3}{8}\ldots\ldots..\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

Result 6:

$I_n = \int tan^n x\, dx$ (n being a positive integer)

$I_n = \int tan^{n-2} x\; tan^2 x\, dx$

$= \int tan^{n-2} x\,(sec^2 x - 1) dx$

$= \int tan^{n-2} x\; d(\tan x) - \int tan^{n-2}\, dx$

$= \frac{tan^{n-1} x}{n-1} - I_{n-2}$

(i) When n is even the ultimate integral is $\int dx = x$

(ii) When n is odd, the ultimate integral is $\int \tan x = \log \sec x$

[$\because \int \tan x\, dx = \int \frac{sin\, x}{cos\, x} dx = -\int \frac{dy}{y}$ putting y = cos x; dy =-sin x

= - log y = log cos x = log (sec x)]

Example 12:

$\int tan^4 x\, dx$

Solution:

$\int tan^4 x\, dx = \frac{tan^3 x}{3} - \int \tan^2 x\, dx$

By putting n = 4 in the formula for I_n

$= \frac{tan^3 x}{3} - \int (sec^2 x - 1)\, dx$

$= \frac{tan^3 x}{3} - tan\, x + x$

Example 13:

$$\int_0^{\pi/4} tan^3 x\; dx$$

Solution:

$\int_0^{\pi/4} tan^3 x\; dx = \left[\frac{tan^2 x}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} tan\, x\; dx$

Put n = 3, $\Rightarrow \frac{1}{2} + [\log \cos x]_0^{\pi/4} = \frac{1}{2} + \log \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \log 2)$

Result 7:

$I_n = \cot^n x$ (n being a positive integer)

$\int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$

$= \int \cot^{n-2} x \cot^2 x\, dx$

$= \int \cot^{n-2} x\, (\text{coses}^2 x - 1)\, dx$

$= \int \cot^{n-2} x\, d(-\cot x) - \int \cot^{n-2} x\, dx$

$= -\frac{\cot^{n-2} x}{n-1} - I_{n-2}$

The ultimate integral is $\int dx$ (or) $\int \cot x$

(i.e.) x (or) log sin x according as n is even (or) odd.

Result 8:

$I_n = \int \sec^n x\, dx$ (n being a positive integer)

$\int \sec^n x\, dx = \int \sec^{n-2} x\, dx\, (\tan x)$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x\, (\sec^2 x - 1)\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^2 x\, dx + (n-2) \int \sec^{n-2} x\, dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

(i) If n be an odd integer, the ultimate integral is

$\int \sec x\, dx = \log(\tan x + \sec x)$

[$\because \int \sec x\, dx = \int \frac{\sec x\,(\sec c + \tan x)}{\sec c\, x + \tan x}\, dx$

y = sec x + tan x, dy = sec x (sec x+ tan x) dx $= \int \frac{dy}{y}$ =log y = log (sec x + tan x)]

(ii) If n be an even integer the ultimate integral is $\int dx = x$

Example 14:

$I = \int \sec^3 x\, dx$

Solution:

Let $\int \sec^3 x\, dx = \int \sec^2 x \sec x\, dx$

$= \int \sec x\, d(\tan x)$

$= \sec x \tan x - \int \tan^2 x \sec x\, dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$

` $= \sec x \tan x - I + \log(\sec x + \tan x)$

$\therefore 2I = \sec x \tan x + \log(\sec x + \tan x)$

Example 15:

$\int \sec^6 x\, dx$

Solution:

$\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx$

$= \sec^4 x\, d(\tan x)$

Where t = tan x

$= \int (1-t^2)^2\, dt$

$= \int (1+2t^2+t^4)\, dt$

$= t + \frac{2t^3}{3} + \frac{t^5}{5}$

$\int \sec^6 x\, dx = \tan x + \frac{2\tan^3}{x} + \frac{\tan^5 x}{5}$

Result 9:

$I_n = \int \mathrm{cosec}^n x\, dx$ (n being or positive integer)

$I_n = \int \mathrm{cosec}^n x\, dx = -\int \mathrm{cosec}^{n-2} x\, dx\, d(\cot x)$

$= - \text{cosec}^{n-2}x \cot x - (n-2) \int \text{cosec}^{n-2}x \cot^2 x \, dx$

$= - \text{cosec}^{n-2}x \cot x -(n-2) \int \text{cosec}^{n-2}x. (\text{cosec}^2 x -1) \, dx$

$= - \text{cosec}^{n-2}x \cot x - (n-2) I_n+(n-2) I_{n-2}$

$\therefore (n-1) I_n = -\text{cosec}^{n-2}x \cot x +(n-2) I_{n-2}$

(i) If n be an odd integer, the ultimate integral is

$\int \text{cosec } x \, dx = - \log (\text{cosec } x + \cot x)$

$[\because \int \text{cosec } x \, dx = \int \frac{\text{cosec } x (\text{cosec } x + \cot x)}{(\text{cosec } x + \cot x)} dx$

$= -\int \frac{d(\text{cosec } x + \cot x)}{\text{cosec } x + \cot x}$ $= - \log (\text{cosec } x + \cot x)]$

(ii) If n be an even integer, ultimate integral is $\int dx = x$

Example 16:

$\int \text{cosec}^4 x \, dx$

Solution:

$\int \text{cosec}^4 x \, dx = \int \text{cosec}^2 x \, \text{cosec}^2 x \, dx$

$= - \int \text{cosec}^2 x \, d(\cot x)$

$= - \int (1+y^2) \, dy$, where $y= \cot x$; $\cot^2 x = \text{cosec}^2 x -1$; $1+\cot^2 x = \text{cosec}^2 x$

$= - y - \frac{y^3}{3}$

$= -\cot x - \frac{\cot^3 x}{3}$

Example 17:

$\int \text{cosec}^5 x \, dx$

Putting n = 5 in the above formula for I_n

$I_n = \frac{- \text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

$n = 5 \Rightarrow \int \text{cosec}^5 x \, dx = \frac{-\text{cosec}^3 x \cot x}{4} + \frac{3}{4} \int \text{cosec}^3 x \, dx$

$$= \frac{-cosec^3x\ cot\ x}{4} - \frac{3}{8}\int cosec^3x\ dx - \frac{3}{8}(\log(cosec\ x + cot\ x))$$

[∵ Let $I = \int cosec^3x\, dx$

$I = \int cosec\ x.\ cosec^2x\ dx$

Applying integrating by parts rule.

$I = cosec \int cosec^2x\ dx - \int[\frac{d}{dx}cosec\ x \int cosec^2x]\ dx$

$I = cosec\ (-cot\ x) - \int cosec\ x\ cot\ x.\ cot\ x\ dx + c$

$I = -\ cosec\ x\ cot\ x - \int cosec\ x\ cot^2x + c$

$∵ cosec^2x - cot^2x = 1$

$I = -cosec\ x.\ cot\ x - \int cosec\ x\ (cosec^2x - 1)\ dx + c$

$I = -\ cosec\ x.\ cot\ x - \int cosec^3x\ dx + \int cosec\ x\ dx + c$

$2I = -\ cosec\ x.\ cot\ x + \log|cosec\ x - cot\ x| + c$

$I = \frac{1}{2}[-\ cosec\ x.\ cot\ x + \log|cosec\ x - cot\ x| + c$

1.3.Integration of Logarithmic Functions:

Result 10:

$I_{m,n} = \int x^m (\log x)^n\, dx$ (where m and n are positive integers)

Hence (or) otherwise evaluate $\int x^4(\log x)^3\ dx$

Solution:

$I_{m,n} = \int(\log x)^n\ d\ (\frac{x^{m+1}}{m+1})$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m(\log x)^{n-1}dx$

$= (\log x)^n\frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$

The ultimate integral is $I_{m,0} = \int x^m\ dx = \frac{x^{m+1}}{m+1}$

$\int(\log x)^3 x^4 dx = \int(\log x)^3 d(\frac{x^5}{5})$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int(\log x)^2 x^4 dx$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int(\log x)^2 d(\frac{x^5}{5})$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\int x^4(\log x)dx$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\{\frac{x^5}{5}\log x - \frac{x^5}{5}\}$

$= x^5\{\frac{1}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 + \frac{6}{125}\log x - \frac{6}{625}\}$

Result 11:

$\int e^{ax}\cos bx\, dx$, a and b are constants.

Let C = $\int e^{ax}\cos bx\, dx$

S = $\int e^{ax}\sin bx\, dx$

C+iS = $\int e^{ax}(\cos bx + i\sin bx)\, dx$

$= \int e^{ax}e^{ibx}dx$ (by Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$)

$= \int e^{x(a+ib)}dx = \frac{e^{x(a+ib)}}{(a+ib)}$

$= e^{ax}\frac{(a-ib)e^{ibx}}{(a+ib)(a-ib)}$

$= e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{(a^2+b^2)}$

C = Real part of $e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{(a^2+b^2)}$

C = $e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{(a^2+b^2)}$

S = Imaginary part of $e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{(a^2+b^2)}$

S = $e^{ax}.\frac{a\sin bx - b\cos bx}{(a^2+b^2)}$

Example 18:

$\int e^{2x}\cos 3x\,dx$

Solution:

C = $\int e^{2x}\cos 3x\,dx$ = Real Part of $\int e^{2x}e^{i3x}dx$

= Real Part of $\int e^{x(2+3i)}dx$

= Real Part of $\frac{e^{x(2+3i)}}{2+3i}$

= Real Part of $\frac{e^{2x}}{13}(2-3i)(\cos 3x + i\sin 3x)$

$= \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x)$

Example 19:

$\int e^{-x} sin^2 x dx$

Solution:

$\int e^{-x} sin^2 x dx = \int e^{-x}\left(\frac{1-\cos 2x}{2}\right)dx = -\frac{e^{-x}}{2} - \frac{1}{2}\int e^{-x}\cos 2x\,dx$

$= -\frac{e^{-x}}{2} - \frac{1}{2}\left(\frac{-e^{-x}\cos 2x + 2\sin 2x}{5}\right)$

Example 20:

$\int e^{ax}\cos mx\cos nx\,dx$

Solution:

$\int e^{ax}\cos mx + \cos nx\,dx = \frac{1}{2}\int e^{ax}\{\cos(m+n)x + \cos(m-n)x\}dx$

a = a, b = m + n

$= \frac{1}{2}e^{ax}\left\{\frac{a\cos(m+n)x + (m+n)\sin(m+n)x}{a^2+(m+n)^2} + \frac{a\cos(m-n)x + (m-n)\sin(m-n)x}{a^2+(m-n)^2}\right\}$

Exercises Problem:

Problem 1:

If $I_n = \int_0^{\pi/2} x^n \ sin\ x\ dx$, n being positive integer, prove that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Solution:

Let $I_n = \int_0^{\pi/2} x^n \ sin\ x\ dx$

$= [\ x^n(-\cos x)]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1}\,(-\cos x)\ dx$

$= n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x\ dx$

$= n[\ x^{n-1}(\sin x)]_0^{\frac{\pi}{2}} - n(n-1)[\int_0^{\frac{\pi}{2}} x^{n-2}\,(\sin x)\ dx\ ]$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\ I_{n-2}$

$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Problem 2:

Evaluate $\int_0^{\pi/2} x^5 \ sin\ x\ dx$

Solution:

Let $I_n = \int_0^{\pi/2} x^n \ sin\ x\ dx$

$\therefore I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\ I_{n-2}$

Putting n=5, we have,

$I_5 = 5\ (\frac{\pi}{2})^4 - 20\ I_3$

Again putting n=3, we have,

$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6\ I_1$

Now, $I_1 = \int_0^{\pi/2} x \ \sin x \ dx$

$= [- x \cos x + \sin x]_0^{\frac{\pi}{2}} = 1$ (Integrating by parts)

$\therefore I_5 = 5(\frac{\pi}{2})^4 - 20 I_3$

$= 5\left(\frac{\pi}{2}\right)^4 - 20(3\left(\frac{\pi}{2}\right)^2 - 6 I_1)$

$= 5\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120 I_1$

$= 5\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120$

$= 120 - 15\pi^2 + \frac{5}{16}\pi^4.$

Problem 3:

Establish a Reduction Formula for $\int_0^{\pi/2} x^n \ \cos x \ dx$ hence find $\int_0^{\pi/2} x^3 \ \cos x \ dx$

Solution:

$I_n = \int_0^{\pi/2} x^n \ \cos x \ dx$ ....(1)

$I_n = \{ x^n (\sin x)_0^{\frac{\pi}{2}}\} - \int_0^{\pi/2} \sin x \ n \ x^{n-1} \ dx$

$= \left(\frac{\pi}{2}\right)^n$ -sin $(\frac{\pi}{2}) - 0 - n\int_0^{\pi/2} x^{n-1} \sin x \ dx$

$= \left(\frac{\pi}{2}\right)^n - n \ \{[x^{n-1} (-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} -\cos x \ (n-1) \ x^{n-2} \ dx \}$

$I_n = \left(\frac{\pi}{2}\right)^n - n \ \{ -\left(\frac{\pi}{2}\right)^{n-1} (-\cos\frac{\pi}{2}) - \int_0^{\pi/2} -\cos x \ (n-1) \ x^{n-2} \ dx \}$

$I_n = \left(\frac{\pi}{2}\right)^n - n \ \{0 + (n-1) I_{n-2} \}$

$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$

$I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n$ ....(2)

Put n=3 in (1)

$\Rightarrow I_3 = \int_0^{\pi/2} x^3 \ \cos x \, dx$

$= \left(\frac{\pi}{2}\right)^3 - 3\,(3-1)\, I_{3-2}$

$= \left(\frac{\pi}{2}\right)^3 - 3(2)\, I_1$

$= \left(\frac{\pi}{2}\right)^3 - 6(\frac{\pi}{2} - 1)$

$I_3 = \left(\frac{\pi}{2}\right)^3 - 3\pi + 6$

$[\therefore \Rightarrow I_1 = \int_0^{\pi/2} x \ \cos x \, dx$

$= [\, x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \sin x \ dx$

$= \frac{\pi}{2} - 0 - (-\cos x)_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - (0+1) = \frac{\pi}{2} - 1]$

Problem 4:

Establish a Reduction Formula for $\int x^n \sin ax \, dx$ hence find $\int_0^{\pi/2} x^3 \ \sin x \, dx$

Solution:

Let $I_n = \int x^n \sin ax \, dx$

$= x^n \left(\frac{-\cos ax}{a}\right) - n \int x^{n-1} \left(\frac{-\cos ax}{a}\right) dx$ (Integrating by parts)

$= \frac{-1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$

$= \frac{-1}{a} x^n \cos ax + \frac{n}{a} \left[x^{n-1} \left(\frac{\sin ax}{a}\right) - (n-1)\int x^{n-2} \left(\frac{\sin ax}{a}\right) dx \right]$

$= \frac{-1}{a} x^n \cos ax + \frac{n}{a^2} x^{n-1} \sin ax - \frac{n(n-1)}{a^2}\int x^{n-2} \sin ax \, dx$

$= \frac{-1}{a} x^n \cos ax + \frac{n}{a^2} x^{n-1} \sin ax - \frac{n(n-1)}{a^2} I_{n-2}$

$a^2 I_n = - a x^n \cos ax + n x^{n-1} \sin ax - n(n-1) I_{n-2}$

To prove: $\int_0^{\pi/2} x^3 \ \sin x \, dx$

Let $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$.

$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3(2)\, I_{3-2}$.

$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6\, I_1$

$I_1 = \int_0^{\pi/2} x \ \sin x \, dx = [- x \ \cos x + \sin x]_0^{\pi/2} = 1$

$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6$

Problem 5:

If $U_n = \int_0^a x^n e^{-x} dx$ Prove that $U_n - (n+a)U_{n-1} + a(n-1)U_{n-2} = 0$

Solution:

$U_n = \int_0^a x^n e^{-x} dx$

$= \int_0^a x^n d(-e^{-x})$

$= [-x^n e^{-x}]_0^a + n\int_0^a e^{-x} x^{n-1} dx$

$= -a^n e^{-a} + n\, U_{n-1}$

Thus $U_n - U_{n-1} = -a^n e^{-a}$ ……….(1)

Similarly $U_{n-1} = -a^n e^{-a} + (n-1)U_{n-2}$

Hence $aU_{n-1} = -a^n e^{-a} + a(n-1)U_{n-2}$ ………..(2)

Equation (1) – (2) gives,

$U_n - (n+a)\, U_{n-1} + a(n-1)\, U_{n-2} = 0$.

Problem 6:

If $\int_0^{\pi/2} \cos^m x \cos n x \, dx$ = f(m,n), Prove that f(m,n) $= \frac{m}{m+n} f(m-1, n-1)$. Hence prove that f(m,n) $= \frac{m}{2^{n+1}}$.

Solution:

$$\int cos^m x \cos nx\, dx = \int cos^m x d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{cos^m x \sin nx}{n} + \frac{m}{n}\int cos^{m-1}x\,.\sin x\,.\sin nx\, dx$$

$$= \frac{cos^m x \sin nx}{n} + \frac{m}{n}\int cos^{m-1}x\{\cos(\mathrm{n}-1)\,x - \cos nx \cos x\,\}dx$$

$$= \frac{cos^m x \sin nx}{n} + \frac{m}{n}\int cos^{m-1}x\,.\cos(\mathrm{n}-1)\,x\,.dx - \frac{m}{n}\int cos^m x\,.\cos \mathrm{n}\,x\,.dx$$

Hence f(m, n) = $\int_0^{\pi/2} cos^m x \cos\mathrm{n}\, x\, dx$

$$= \frac{1}{m+n}\left\{(cos^m x \sin nx)_0^{\pi/2} + \frac{m}{m+n}\int_0^{\pi/2} cos^{m-1}x \cos(\mathrm{n}-1)\,x\, dx\right\}$$

$= \frac{m}{m+n}\, f(m-1, n-1)$ as the first term vanishes at both limits.

Putting m = n,

f(m, n) = $\frac{1}{2}\, f(n-1, n-1) = \frac{1}{2^2}\, f(n-2, n-2) = \frac{1}{2^n}\, f(0,0)$

By the repeated application of the same formula,

$$= \frac{1}{2^n}\int_0^{\pi/2} dx = \frac{\pi}{2^{n+1}}$$

Problem 7:

If $\int_0^{\pi/2} cos^m x \sin \mathrm{n}\, x\, dx$ = f(m,n), Prove that f(m, n) = $\frac{1}{m+n} + \frac{m}{m+n}\, f(m-1, n-1)$, Hence deduce that f(m,n) = $\frac{1}{2^{n+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{2} + \frac{2^4}{2} + \cdots + \frac{2^m}{m}\right]$

Solution:

f(m, n) = $\int_0^{\pi/2} cos^m x \sin \mathrm{n}\, x\, dx$

$$= \frac{1}{n}\int_0^{\pi/2} cos^m x d(-\cos nx)$$

$$= \left[-cos^m x\left(\frac{\cos nx}{n}\right)\right]^{\pi/2} - \frac{m}{n}\int_0^{\pi/2} cos^{m-1}x\,.\sin x\,.\cos nx\, dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} cos^{m-1}x\,[\sin nx.\cos x - \sin[(n-1)x]]\, dx$$

[since $\sin(nx - x) = \sin nx \cos x - \cos nx.\sin x$]

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} cos^m x \,.\sin x\, dx + \frac{m}{n}\int_0^{\pi/2} cos^{m-1} x \sin[(n-1)x]\, dx$$

$$= \frac{1}{n} - \frac{m}{n} f(m,n) + \frac{m}{n} f(m-1, n-1)$$

$$\therefore \left(1 + \frac{m}{n}\right) f(m,n) = \frac{1}{n} + \frac{m}{n} f(m-1, n-1)$$

$$\left(\frac{m+n}{n}\right) f(m,n) = \frac{1}{n} + \frac{m}{n} f(m-1, n-1)$$

$$\therefore\ f(m,n) = \frac{1}{m+n} + \frac{m}{m+n}\ f(m-1, n-1)$$

Now,

$$f(m,m) = \frac{1}{2m} + \frac{m}{2m}\ f(m-1, m-1)$$

$$= \frac{1}{2m} + \frac{1}{2} f(m-1, m-1)$$

$$= \frac{1}{m} + \frac{1}{2}\left[\frac{1}{2m-2} + \frac{1}{2}\ f(m-2, m-2)\right]$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\left[\frac{1}{2m-4} + \frac{1}{2} f(m-3, m-3)\right]$$

…………………………

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-1)} + \cdots + \frac{1}{2^{m-1}} f(1,1)$$

Now, $f(1,1) = \int_0^{\pi/2} \cos x \sin x\, dx$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx$$

$$= \left[\frac{-\cos 2x}{4}\right]_0^{\pi/2} = \frac{1}{2}$$

$$\therefore\ f(m,n) = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-1)} + \cdots + \frac{1}{2^{m-1}}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^{m+1}}\left[\frac{2^m}{m} + \frac{2^{m-1}}{m-1} + \cdots + \frac{2}{1}\right]$$

$$f(m,n) = \frac{1}{2^{m+1}}\left[\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^m}{m}\right]$$

Problem 8:

If $I_n = \int \frac{dx}{(x^2+1)^n}$, Show that $2nI_{n+1} = (2n-1)\, I_n + \frac{x}{(x^2+1)^n}$. Hence find $\int_0^1 \frac{x}{(x^2+1)^2} dx$.

[Hint: Put $x = \tan\theta$]

Solution:

$$I_n = \int \frac{x\,dx}{(x^2+1)^n}$$

$$I_n = \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2dx}{(x^2+1)^{n+1}}$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{1+x^2-1dx}{(x^2+1)^{n+1}}$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{dx}{(x^2+1)^n} - 2n \int \frac{dx}{(1+x^2)^{n+1}}$$

$$\therefore I_n = \frac{x}{(x^2+1)^n} + 2nI_n - 2nI_{n+1}$$

$$\therefore 2nI_{n+1} = (2n-1)I_n + \frac{x}{(x^2+1)^n}$$

$$\Longrightarrow \int_0^1 \frac{x\,dx}{(x^2+1)^2}$$

Put $x = \tan\theta$

$\Longrightarrow dx = sec^2\theta\, d\theta$

x = 0 $\Longrightarrow \theta = 0$

x = 1 $\Longrightarrow \theta = \frac{\pi}{4}$

$$\Longrightarrow \int_0^1 \frac{\tan\theta}{(\tan^2\theta+1)^2}\ \text{Sec}^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{(\text{Sec}^2\theta)^2}\ \text{Sec}^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta \tan\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta}{2} d\theta$$

$$= \frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}[-\cos 2(\frac{\pi}{4}) + \cos 0]$$

$$= \frac{1}{4}\,[0+1] = \frac{1}{4}$$

Problem 9:

If $I_n = \int_0^{\pi/2} \theta \text{Sin}^n \theta d\theta$ and $n > 1$, Prove that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$. Deduce that $I_5 = \frac{149}{225}$

Solution:

$I_n = \int (\sin\theta)^{n-1}(\theta \sin\theta) d\theta$

Taking $u = (\sin\theta)^{n-1}$

du = $\theta \sin\theta \, d\theta$

we get v = - $\theta \cos\theta + \sin\theta$

$$\therefore I_n = [(\sin\theta - \theta\cos\theta)(\sin\theta)^{n-1}]_0^{\frac{\pi}{2}}$$

$$-(n-1)\int_0^{\pi/2} (\sin\theta - \theta\cos\theta)(\sin\theta)^{n-2} \cos\theta \, d\theta$$

$$= 1 - (n-1)\int_0^{\pi/2} \sin^{n-1}\theta \cos\theta \, d\theta + (n-1)\int_0^{\pi/2} \theta(\sin\theta)^{n-2} \ cos^2\theta \, d\theta$$

$$= 1 - (n-1)\int_0^{\pi/2} (\sin\theta)^{n-1} d(\sin\theta) + (n-1)\int_0^{\pi/2} \theta(\sin\theta)^{n-2} (1 - sin^2\theta) \, d\theta$$

$$= 1 - \frac{n-1}{n}[\sin^n\theta]_0^{\pi/2} + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n\big(1 + (n-1)\big) = 1 - \left(\frac{n-1}{n}\right) + (n-1)\, I_{n-2}$$

$$\therefore nI_n = \frac{1}{n} + (n-1)\, I_{n-2}$$

$\therefore I_n = \frac{1}{n^2} + \left(\frac{n-1}{n}\right) I_{n-2}$

Now, $I_5 = \left(\frac{4}{5}\right) I_3 + \frac{1}{25}$

$I_3 = \left(\frac{2}{3}\right) I_1 + \frac{1}{9}$

Also, $I_n = \int_0^{\pi/2} \theta \sin\theta \, d\theta$

$= [-\theta\cos\theta]_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \, d\theta = [-\theta\cos\theta + \sin\theta]_0^{\pi/2} = 1$

Here $I_3 = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

$I_5 = \frac{4}{5}\left(\frac{7}{9}\right) + \frac{1}{25}$

$= \frac{28}{45} + \frac{1}{25}$

$= \frac{700+45}{1125}$

$= \frac{745}{1125} = \frac{149}{225}$

Problem 10:

If $I_n = \int_0^{\pi/2} x\cos^n x dx$ where $n > 1$, Show that $I_n = \frac{-1}{n^2} + \frac{n-1}{n} I_{n-2}$.

Solution:

$I_n = \int_0^{\pi/2} x\cos^n x dx$ .... (1)

$$I_n = \int_0^{\pi/2} x\cos^{n-1}x \, cos \, xdx$$

$u = x\cos^{n-1}x$; $dv = cos \, xdx$

$$I_n = [\mathrm{x}\cos^{n-1}x \ \sin x]_0^{\pi/2} - \int_0^{\pi/2} (x(n-1)\cos^{n-2}x(-\sin x) + \cos^{n-1}x.1) \sin x \, dx$$

$$I_n = 0 - \int_0^{\pi/2} -(n-1)x\cos^{n-2}x\sin^2 xdx + \cos^{n-1}x.\sin xdx$$

$$I_n = (n-1)\int_0^{\pi/2} x\cos^{n-2}x(1-\cos^2 x)dx - \int_0^{\pi/2} \cos^{n-1}x \sin x\, dx$$

$$I_n = (n-1)\int_0^{\pi/2} (x\cos^{n-2}x - x\cos^n x)dx - \int_0^{\pi/2} t^{n-1}dt$$

since $t = \cos x$; $dt = -\sin x\, dx$

$$I_n = (n-1)\int_0^{\pi/2} x\cos^{n-2}x - (n-1)\int_0^{\frac{\pi}{2}} x\cos^n x dx + \left[\frac{t^n}{n}\right]_0^{\pi/2}$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n + \left[\frac{1}{n}\right]$$

$$I_n + (n-1)I_n = (n-1)I_{n-2} - \frac{1}{n}$$

$$I_n = \frac{-1}{n^2} + \frac{n-1}{n} I_{n-2}$$

Problem 11:

Integrate $e^x \sin 2x$

Solution:

$$\int e^{ax}\sin bx\, dx = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2}$$

a = 1, b = 2

$$\int e^x \sin 2x\, dx = e^x \frac{\sin 2x - 2\cos 2x}{1^2+2^2} = \frac{e^x}{5}[\sin 2x - 2\cos 2x]$$

Problem 12:

Integrate $e^{-3x}\sin\frac{x}{2}$

Solution:

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int e^{-3x}\sin\frac{x}{2}dx = \frac{e^{-3x}}{(-3)^2+(1/2)^2}\left[-3\sin\frac{x}{2} - \frac{1}{2}\cos\frac{x}{2}\right]$$

$$\int e^{-3x} \sin\frac{x}{2} dx = \frac{e^{-3x}}{9 + (1/4)} \left[-3\sin\frac{x}{2} - \frac{1}{2}\cos\frac{x}{2}\right]$$

$$\int e^{-3x} \sin\frac{x}{2} dx = \frac{-4e^{-3x}}{37} \left[3\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right]$$

Problem 13:

Integrate $e^{2x} \cos(3x + 4)$

Solution:

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a\cos bx + b \sin bx]$$

Here a = 2, b = 3

$$\int e^{2x} \cos(3x + 4) \, dx = \frac{e^{2x}}{2^2 + 3^2} [2\cos(3x + 4) + 3 \sin(3x + 4)x]$$

$$\int e^{2x} \cos(3x + 4) \, dx = \frac{e^{2x}}{13} [2\cos(3x + 4) + 3 \sin(3x + 4)x]$$

Problem 14:

Integrate $e^{-3x} \sin 3x \sin 2x$

Solution:

2 sin A sin B = cos(A-B) - cos(A+B)

$$\sin 3x \sin 2x = \frac{1}{2}[\cos(3 - 2x) - \cos(3 + 2x)]$$

$$= \frac{1}{2}[\cos\ x - \cos 5x]$$

$$\int e^{-3x} \sin 3x \sin 2x \, dx = \int e^{-3x} \frac{1}{2}[\cos\ x - \cos 5x] dx$$

$$= \frac{1}{2}\left[\int e^{-3x} \cos x\ \ dx - \int e^{-3x} \cos 5x\ \ dx\right]$$

$$= \frac{1}{2}\left\{\frac{e^{-3x}}{(-3)^2+1^2}\ (-3\cos x + \sin x) - \frac{e^{-3x}}{(-3)^2+5^2}\ (-3\cos 5x + 5\sin 5x)\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{-3x}}{10}\ (-3\cos x + \sin x) - \frac{e^{-3x}}{34}\ (-3\cos 5x + 5\sin 5x)\right\}$$

$$= \frac{e^{-3x}}{4}\left\{\frac{\sin x - 3\cos x}{5} + \frac{3\cos 5x - 5\sin 5x}{5}\right\}$$

Problem 15:

Integrate $e^{4x}\cos 3x$

Solution:

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\ [a\cos bx + b\sin bx]$$

Here a = 4, b = 3

$$\int e^{4x}\cos 3x\, dx = \frac{e^{4x}}{4^2+3^2}\ [4\cos 3x + 3\sin 3x]$$

$$= \frac{e^{4x}}{25}[4\cos 3x + 3\sin 3x]$$

Problem 16:

Integrate $e^{ax}\sin(bx+c)$

Solution:

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\ [a\sin bx - b\cos bx]$$

Here a = a, b = b

$$\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\ [a\sin(bx+c) - b\cos(bx+c)]$$

$$= \frac{e^{ax}}{a^2+b^2}\ [a\sin(bx+c) - b\cos(bx+c)]$$

Problem 17:

Integrate $e^{x}cos^2 xdx$

Solution:

$$\int e^x cos^2 x dx = \int e^x \left(\frac{\cos 2x + 1}{2}\right) dx = \frac{1}{2}\int e^x(\cos 2x + 1)dx$$

$$= \frac{1}{2}[\int e^x \cos 2x\, dx + \int e^x dx]$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\ [a\cos bx + b \sin bx]$$

$$\int e^x \cos 2x\, dx = \frac{e^x}{1^2+2^2}\ [\cos 2x + 2 \sin 2x]$$

$$= \frac{1}{2}\left[\frac{e^x}{5}(\cos 2x + 2 \sin 2x) + e^x\right]$$

$$\int e^x cos^2 x dx = \frac{e^x}{2}\left[1 + \frac{\cos 2x + 2 \sin 2x}{5}\right]$$

Exercises 1:

1.Evaluate $\int (\log x)^3\, x^2\, dx$ and $\int (\log x)^5\, x^5\, dx$

2. Evaluate (i) $\int sin^5 x\, dx$ (ii) $\int_0^{\frac{\pi}{2}} sin^4 x\, dx$

3. Evaluate (i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2\theta\, d\theta\, cos^7\theta\, d\theta$ (ii) $\int_0^{\frac{\pi}{2}} sin^2\theta\, (sin^3\theta + cos^3\theta\,)d\theta$

4. Evaluate (i) $\int tan^6\, x\, dx$ (ii) $\int_0^{\frac{\pi}{2}} tan\,^5 x\, dx$

5. Evaluate (i) $\int cot^4\, x\, dx$ (ii) $\int sec^4\, x\, dx$ (iii) $\int cosec^3\, x\, dx$

6. Integrate (i) $e^x\ sin3x \cos 2x$ (ii) $e^{2x} \cos 5\, x \cos 4x$

1.4.Bernoulli's Formula

Theorem :1 (Integrating by parts)

Let u and v be differentiable function of x. Then $\int udv = uv - \int v\, du.$

Proof:

We know that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrating we get,

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$= \int u\,dv + \int v\,du$$

$$\therefore \int udv = uv - \int v\,du$$

Note: The method of evaluating a given integral by using the above theorem is called integration by parts. In applying this method, we must choose u and v carefully so that the resulting integral is simplex than the given integral.

Theorem: 2 (Bernoulli's Formula)

Let u and v be differentiable function of x. Suppose there exists a positive integer n such that $u^{(n)} = 0$, then $\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots + (-1)^n u^{(n)} v_n$

where $v_1 = \int v\,dx$; $v_2 = \int v_1 dx$

Proof:

$\int udv = uv - \int v\,du$ (by theorem 2.1)

$$= uv - \int u'\,d(v_1)$$

$$= uv - u'v_1 + \int v_1 du'$$

$$= uv - u'v_1 + \int u''d(v_2)$$

$$= uv - u'v_1 + u''v_2 - \int v_2 du''$$

Proceeding like this we get the required formula

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Example 1:

Evaluate $\int x^4 e^x dx$

Here $u = x^4$ $\quad dv = e^x dx$

$u' = 4x^3$ $\quad v = e^x$

$u'' = 12x^2$ $\quad v_1 = e^x$

$$u''' = 24x \qquad v_2 = e^x$$

$$u'''' = 24 \qquad v_3 = e^x$$

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int x^4e^xdx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x - \cdots$$

Example 2:

Evaluate $\int x^3 \cos 2x \, dx$

Solution:

$$\int x^3 \cos 2x \, dx$$

Here $u = x^3$ $dv = \cos 2x \, dx$

$$u' = 3x^2 \qquad v = \frac{\sin 2x}{2}$$

$$u'' = 6x \qquad v_1 = \frac{-\cos 2x}{4}$$

$$u''' = 6 \qquad v_2 = \frac{-\sin 2x}{8}$$

$$v_3 = \frac{\cos 2x}{16}$$

$$\int x^3 \cos 2x \, dx = \frac{x^3 \sin 2x}{2} - 3x^2\left(\frac{-\cos 2x}{4}\right) + 6x\left(\frac{-\sin 2x}{8}\right) - 6\left(\frac{\cos 2x}{16}\right)$$

$$= \frac{1}{2}\left[x^3 \sin 2x + \frac{3x^2\cos 2x}{2} - \frac{3x\sin 2x}{2} - \frac{3\cos 2x}{4}\right]$$

Exercises Problem:

Problem 1:

Integrate x^3e^{-2x}

Solution:

$$u = x^3 \qquad dv = e^{-2x}dx$$

$u' = 3x^2 \qquad v = \frac{-e^{-2x}}{2}$

$u'' = 6x \qquad v_1 = \frac{e^{-2x}}{4}$

$u''' = 6 \qquad v_2 = -\frac{e^{-2x}}{8}$

$v_3 = \frac{e^{-2x}}{16}$

By Bernoulli's formula

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int x^3 e^{-2x} dx = x^3\left(\frac{-e^{-2x}}{2}\right) - 3x^2\left(\frac{e^{-2x}}{4}\right) + 6x\left(-\frac{e^{-2x}}{8}\right) - 6\left(\frac{e^{-2x}}{16}\right) + \cdots$$

$$= \frac{1}{8} e^{-2x}[-4x^3 - 6x^2 - 6x - 3]$$

$$= \frac{-1}{8} e^{-2x}[4x^3 + 6x^2 + 6x + 3]$$

Problem 2:

Integrate $\int x^4 \sin x \, dx$

Solution:

$$\int x^4 \sin x \, dx$$

Here $u = x^4$ $dv = \sin x \, dx$

$u' = 4x^3 \qquad v = -\cos x$

$u'' = 12x^2 \qquad v_1 = -\sin x$

$u''' = 24x \qquad v_2 = \cos x$

$u'''' = 24 \qquad v_3 = \sin x$

$v_4 = -\cos x$

By Bernoulli's formula.,

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int x^4 \sin x\, dx = x^4(-\cos x) - 4x^3(-\sin x) + 12x^2 \cos x - 24x \sin x \qquad + 24(-\cos x)$$

$$\int x^4 \sin x\, dx = \cos(-x^4 + 12x^2 - 24) + 4x(x^2 - 6) \sin x$$

Problem 3:

Integrate $\int x^3 \sin 3x\, dx$

Solution:

$$\int x^3 \sin 3x\, dx$$

Here $u = x^3$ $dv = \sin 3x\, dx$

$u' = 3x^2$ $\qquad v = -\frac{\cos 3x}{3}$

$u'' = 6x$ $\qquad v_1 = -\frac{\sin 3x}{9}$

$u''' = 6$ $\qquad v_2 = \frac{\cos 3x}{27}$

By Bernoulli's formula.,

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int x^3 \sin 3x\, dx = x^3\left(-\frac{\cos 3x}{3}\right) - 3x^2\left(-\frac{\sin 3x}{9}\right) + 6x\left(\frac{\cos 3x}{27}\right) - 6\left(\frac{\sin 3x}{81}\right)x$$

$\int x^3 \sin 3x\, dx = \frac{1}{27}[3x(2-3x^2)\cos 3x + \sin 3x(9x^2-2)]$

Problem 4:

Integrate $\int x^2(e^x + e^{-x})\, dx$

Solution:

$\int x^2(e^x + e^{-x})\, dx$

Here $u = x^2$ $dv = e^x - e^{-x}$

$u' = 2x$ $v = e^x + e^{-x}$

$u'' = 2$ $v_1 = e^x - e^{-x}$

By Bernoulli's formula.,

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$\int x^2(e^x + e^{-x})\,dx = (x^2)(e^x - e^{-x}) - 2x(e^x - e^{-x}) - 2x(e^x + e^{-x}) + 2(e^x - e^{-x})$

$= x^2e^x - x^2e^{-x} - 2xe^x - 2xe^{-x} + 2e^x - 2e^{-x}$

$= (x^2 - 2x + 2)\,e^x + (-x^2 - 2x - 2)\,e^{-x}$

$= (x^2 - 2x + 2)\,e^x - (x^2 + 2x + 2)\,e^{-x}$

Exercises 2:

1. $\int x^3 \sin nx\, dx$
2. $\int x^5 \cos\frac{x}{2} dx$

UNIT-II

Multiple Integrals - definition of double integrals - evaluation of double integrals – double integrals in polar coordinates - Change of order of integration.

Chapter 2: Sections 2.1-2.5

2.1. Multiple Integrals:

Let $f(x)$ be a continuous function in the closed interval from $x = a\ to\ x = b$. Hence the function is bounded in the interval. Let $b > a$. Divide the interval (a, b) into n sub-intervals $x_1 - a\ , x_2 - x_1, x_3 - x_2 \ldots\ldots b - x_{n-1}$, where $a, x_1, x_2, x_3 \ldots \ldots x_{n-1}\ , b$ are in ascending order of magnitudes. Let ξ_r be any point of the sub-interval (x_{r-1}, x_r). Taking $a = x_0\ and\ b = x_n$

Consider the sum $f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) \ldots \ldots \ldots f(x_n)(x_n - x_{n-1})$.

This sum tends to a definite limit when the number n of the sub-intervals tends to infinity, i.e., the length of each sub-interval tends to zero, as a and b are finite. We have already seen that this limit is called the definite integral of f(x) with respect to x from x=a to x=b and is written as $\int_a^b f(x)dx$. Even in the case of simple functions the evaluation of an integral from this definition is not quite easy. So we evaluate $\int_a^b f(x)dx$ from the following result:

$\int_a^b f(x)dx = F(b) - F(a)$, where $\frac{d}{dx}\ F(x) = f(x)$.

2.2. Definition of the double integral:

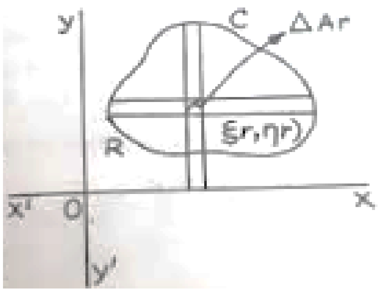

Figure 2.1

Let $f(x, y)$ be a continuous and single valued function of x and y within a region R bounded by a closed curve C upon the boundary C. Let the region R be subdivided in any manner into n sub-regions of area $\Delta A_1,\ \Delta A_2, \ldots \ldots, \Delta A_n$.

Let (ξ_r, η_r) be any point in the sub-region of area ΔA_r and consider the sum $\sum_{r=1}^{n} f(\xi_\rho, \eta_r)\Delta A_r$.

The limit of this sum as $n \to \infty$ and $\Delta A_r \to 0$ $(r = 1, 2, \ldots.)$ is defined as the double integral of $f(x, y)$ over the region R.

Thus $\iint_R f(x, y)dA = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\xi_r, \eta_\rho\right)\Delta A_r$.

The region R is called the region of integration corresponding to interval of integration (a, b) in the case of the single integral. This integral is sometimes written as $\iint_R f(x, y)dxdy$.

2.3. Evaluation of the double integral:

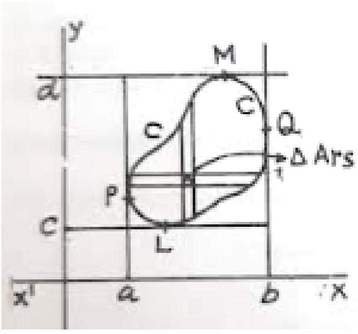

Figure 2.2

$$\int_R f(x, y)dA = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y)dxdy$$

The double integral is evaluated by considering $f(x, y)$ as a function of x alone but regarding y as a constant and integrating it between $x = f_1(y)$ and $f_2(y)$ and then integrating the resulting function of y between $y = c$ and $y = d$.

Similarly, by taking the sum of the terms in each column and then adding these sums.

$$\int_R f(x, y)dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dydx$$

Hence $f(x,y)$ is first considered as a function of y alone and integrated between $\varphi_1(x)$and $\varphi_2(x)$ where the equations of the curves PLQ and PMQ are respectively $y = \varphi_1(x)$ and $\varphi_2(x)$ and then the resulting function of x is integrated between $x = a$ and $x = b$.

Corollary 1:

If the region of integration is a rectangle between the lines $x = a, x = b, y = c, y = d$ then

$$\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$$

$$= \int_a^b \int_c^d f(x,y)dxdy$$

Thus for constant limits, the order of integration is immaterial.

Note:

The integral $\int_a^b \int_{f_1(y)}^{f_2(y)} f(x,y)dxdy$ is the integral over the region bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ for the values of x between a and b. For changing its order, one should sketch the region of integration. From the sketch, the limits of x and y should be determined.

Example 1:

Evaluate $\int\int xy\, dx\, dy$ over the area in the first quadrant bounded by the circle

$x^2 + y^2 = a^2$

Solution:

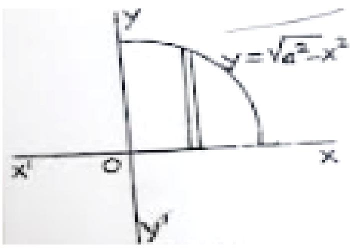

Figure 2.3

If x is constant y varies from o to $\sqrt{a^2 - x^2}$, x varies from o to a

$$\iint xydxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xydydx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\,(a^2 - x^2)dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a dx$$

$$= \frac{1}{2}\left(\frac{a^4}{4}\right)$$

$$= \frac{a^4}{8}$$

Example 2:

Evaluate $\iint (x^2 + y^2)dxdy$ over the region for which x, y are each ≥ 0 and $x + y \leq 1$

Solution:

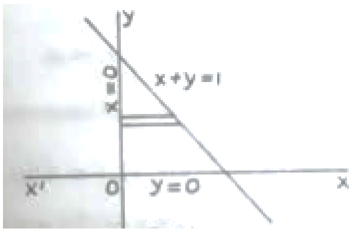

Figure 2.4

If x is constant y varies fro 0 to $1 - x$.

x varies from 0 to 1

$$\iint (x^2 + y^2)dxdy = \int_0^1 \int_0^{1-x} (x^2 + y^2)dydx$$

$$= \int_0^1 [x^2y + \frac{y^3}{3}]_0^{1-x} dx$$

$$= \int_0^1 [x^2\,(1 - x) + \frac{(1-x)^3}{3}]dx$$

$$= \int_0^1 (x^2 - x^3 + \frac{1-3x-3x^2-x^3}{3})dx$$

$$= \int_0^1 (3x^2 - 3x^3 + 1 - 3x - 3x^2 - x^3)/3 \ \ dx$$

$$= \frac{1}{3}\int_0^1 (-4x^3 + 6x^2 - 3x + 1)dx$$

$$= \frac{1}{3}\left[-\frac{4x^4}{4} + \frac{6x^3}{3} - \frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}[-1 + 2 - \frac{3}{2}+1-0]$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)$$

$$\iint (x^2 + y^2)dxdy = \frac{1}{6}$$

2.4. Change of order of integration:

Example 1:

Change the order of integration in the integral $\int_{x=0}^{a} \int_{y=x^2/a}^{2a-x} xy\ dx\ dy$ and evaluate it.

Solution:

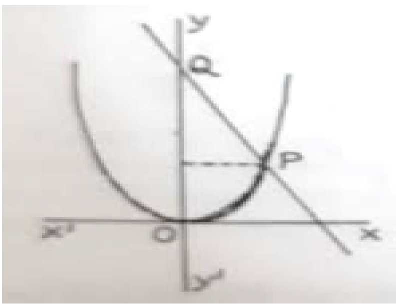

Figure 2.5

Given, x varies from 0 to a

y varies from $\frac{x^2}{a}$ to $2a - x$

(i.e.,) The region is bounded by $x = 0,\ x = a$ and $y = \frac{x^2}{a}, y = 2a - x$

By changing the order of integration we first integrate with respect to x and then with respect to y

Therefore, y varies from 0 to a

x varies o to $\sqrt{ay}$

and y varies from a to 2a

x varies 0 to 2a – y

$$\therefore \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dxdy = \int_0^a \int_0^{\sqrt{ay}} xy\,dxdy + \int_a^{2a} \int_0^{2a-y} xy\,dxdy$$

$$= \int_0^a \left[\frac{x^2y}{2}\right]\begin{matrix}\sqrt{ay}\\0\end{matrix}\; dy + \int_a^a \left[\frac{x^2y}{2}\right]\begin{matrix}2a-y\\0\end{matrix}\; dy$$

$$= \frac{1}{2}\int_0^a ay^2dy + \frac{1}{2}\int_a^{2a} (2a-y)^2 y\,dy$$

$$= \frac{1}{2}\left[\frac{ay^3}{3}\right]\begin{matrix}a\\0\end{matrix} + \frac{1}{2}\int_a^{2a} (4a^2+y^2-4ay)y\,dy$$

$$= \frac{1}{2}\left[\frac{a^4}{3}\right] + \frac{1}{2}\int_a^{2a} (4a^2y+y^3-4ay^2)\,dy$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right]\begin{matrix}2a\\0\end{matrix}$$

$$= \frac{a^4}{6} + \frac{1}{24}[24a^2y^2 + 3y^4 - 16ay^3]\begin{matrix}2a\\0\end{matrix}$$

$$= \frac{a^4}{6} + \frac{1}{24}[160a^4 - 155a^4]$$

$$= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{4a^4 + 5a^4}{24}$$

$$= \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$

Example 2:

By changing the order of integration, evaluate $\int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} \frac{e^{-y}}{y} dxdy$.

Solution:

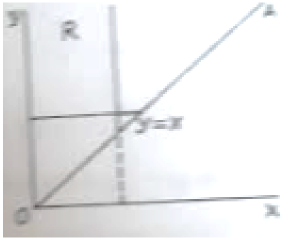

Figure 2.6

Given x varies from 0 to ∞, y varies from x to ∞.

By changing the order of integration, we first integrate with respect to x and with respect to y

x varies from x = 0 to x = y

y varies from y = 0 to $y = \infty$

$$\therefore \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dxdy = \int_{y=0}^{y=\infty} \int_{x=0}^{x=a} \frac{e^{-y}}{y} dxdy$$

$$= \int_0^{\infty} \int_0^{y} \frac{e^{-y}}{y} dxdy$$

$$= \int_0^{\infty} \left[\frac{xe^{-y}}{y}\right]_0^y dy$$

$$= \int_0^{\infty} (e^{-y} + 0)\, dy$$

$$= \int_0^{\infty} e^{-y} dy$$

$$= -[e^{-y}]_0^{\infty}$$

$$= -e^{-\infty} + e^{0}$$

$$= 0 + 1 = 1$$

Exercises Problem:

Problem 1:

Evaluate the following integrals.

(i) $\int_0^a \int_0^b (x^2 + y^2)\, dxdy.$

Solution:

If y is constant x varies from 0 to b

x varies from 0 to a

$$\int_0^a \int_0^b (x^2 + y^2)\, dxdy = \int_0^a \left[\frac{x^3}{3} + y^2 x\right]_0^b dy$$

$$= \int_0^a \left[\frac{b^3}{3} + by^2\right] dy = \int_0^a \left(\frac{b^3 + 3by^3}{3}\right) dy$$

$$= \frac{1}{3}\int_0^a (b^3 + 3by^3)\, dy = \frac{1}{3}\left[b^3 y + \frac{3by^3}{3}\right]_0^a$$

$$= \frac{1}{3}[b^3 y + by^3]_0^a$$

$$= \frac{1}{3}[ab^3 + ba^3]$$

$$= \frac{ab}{3}[b^2 + a^2]$$

$$\int_0^a \int_0^b (x^2 + y^2)\, dxdy = \frac{ab}{3}[b^2 + a^2]$$

(ii) $\int_0^3 \int_1^2 xy(x + y)\, dydx.$

Solution:

If x is constant y varies from 1 to 2.

x varies from 0 to 3.

$$\therefore \int_0^3 \int_1^2 xy(x + y)\, dydx = \int_0^3 \int_1^2 (x^2 y + xy^2)\, dydx = \int_0^3 \left[x^2\left(\frac{y^2}{2}\right) + x\left(\frac{y^3}{3}\right)\right]_1^2 \text{dx}$$

$$= \int_0^3 \left[\frac{4x^2}{2} + \frac{8x}{3} - \frac{x^2}{2} - \frac{x}{3}\right] dx$$

$$= \int_0^3 \left[2x^2 + \frac{8x}{3} - \frac{x^2}{2} - \frac{x}{3}\right] dx$$

$$= \int_0^3 \left(\frac{(12x^2 + 16x - 3x^2 - 2x)}{6}\right) dx$$

$$= \frac{1}{6}\int_0^3 (12x^2 + 16x - 3x^2 - 2x)\, dx$$

$$= \frac{1}{6}\left[12\left(\frac{x^3}{3}\right) + 16\left(\frac{x^2}{2}\right) - 3\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right)\right]_0^3$$

$$= \frac{1}{6}[4x^3 + 8x^2 - x^3 - x^2]_0^3$$

$$= \frac{1}{6}[3x^3 + 7x^2]_0^3$$

$$= \frac{1}{6}[3(3)^3 + 7(3)^2] = \frac{1}{6}[81 + 63] = \frac{1}{6}(144) = 24$$

$$\int_0^3 \int_1^2 xy(x + y)\, dydx = 24$$

iii) $\int_0^a \int_0^b xy(x - y)\, dxdy.$

Solution:

If x is constant, y varies from a to b. X varies from o to a

$$\therefore \int_0^a \int_0^b xy(x - y)\, dxdy = \int_0^a \int_0^b (x^2y - y^2x)\, dydx$$

$$= \int_0^a \left[\frac{x^2y}{2} - \frac{xy^3}{3}\right]_0^b dx$$

$$= \int_0^a \left[\frac{x^2b}{2} - \frac{xb^3}{3}\right] dx$$

$$= \int_0^a \left[\frac{3x^2b^2 - 2xb^3}{6}\right] dx$$

$$= \frac{1}{6}\int_0^a (3x^2b^2 - 2xb^3)\, dx$$

$$= \frac{1}{6}\left[3b^2\left(\frac{x^3}{3}\right) - 2b^3\left(\frac{x^2}{2}\right)\right]_0^a$$

$$= \frac{1}{6}\left[3b^2\left(\frac{a^3}{3}\right) - 2b^3\left(\frac{a^2}{2}\right)\right]$$

$$= \frac{1}{6}[b^2a^3 - b^3a^2]$$

$$\int_0^a \int_0^b xy(x-y)\, dxdy = \frac{a^2b^2(a-b)}{6}$$

iv) $\int_1^2 \int_1^x xy^2 dydx$

Solution:

If x is constant y varies from 1to x , x varies from 1 to 2

$$\int_1^2 \int_1^x xy^2 dydx = \int_1^2 [\frac{xy^3}{3}]_1^x dx$$

$$= \int_1^2 [\frac{xx^3}{3} - \frac{x.1}{3}]\, dx$$

$$= \frac{1}{3}\int_1^2 (x^4 - x)dx$$

$$= \frac{1}{3}\left[\frac{x^5}{5} - \frac{x^2}{2}\right]_1^2 dx$$

$$= \frac{1}{3}\left[\left(\frac{2^5}{5} - \frac{2^2}{2}\right) - \left(\frac{1}{5} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{3}\left[\frac{64-20}{10} + \frac{3}{10}\right]$$

$$= \frac{1}{3}[\frac{44}{10} + \frac{3}{10}]$$

$$= \frac{47}{30}$$

v) $\int_0^a \int_0^x (x^2 + y^2)\, dydx$

Solution:

If x is constant y varies from 0 to x . x varies from 0 to a

$$\int_0^a \int_0^x (x^2 + y^2)\, dydx = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^x dx$$

$$= \int_0^a \left[x^3 + \frac{x^3}{3}\right] dx$$

$$= \int_0^a \left[\frac{3x^3 + x^3}{3}\right] dx$$

$$= \frac{1}{3}\int_0^a [4x^3] dx$$

$$= \frac{1}{3}\left[4\frac{x^4}{4}\right]_0^a$$

$$\therefore \int_0^a \int_0^x (x^2 + y^2)\, dydx = \frac{1}{3}a^4$$

vi) $\int_0^2 \int_{x^2}^{2x} (2x + 3y)\, dydx$

Solution:

If x is constant y varies from x^2 to $2x$. x varies from 0 to 2

$$\int_0^2 \int_{x^2}^{2x} (2x + 3y)\, dydx = \int_0^2 \left[2xy + 3\frac{y^2}{2}\right]_{x^2}^{2x} dydx$$

$$= \int_0^2 \left[2x(2x) + 3\frac{(2x)^2}{2} - \left(2x(x^2) + 3\frac{(x^2)^2}{2}\right)\right] dx$$

$$= \int_0^2 \left[4x^2 + 6x^2 - 2x^3 - 3\frac{x^4}{2}\right] dx$$

$$= \int_0^2 \left[\frac{8x^2 + 12x^2 - 4x^3 - 3x^4}{2}\right] dx$$

$$= \frac{1}{2}\int_0^2 [8x^2 + 12x^2 - 4x^3 - 3x^4] dx$$

$$= \frac{1}{2}\left[8\frac{x^3}{3} + 12\frac{x^3}{3} - 4\frac{x^4}{4} - 3\frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{20x^3}{3} + \frac{4x^4}{4} - \frac{3x^5}{5}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{20(2)^3}{3} + \frac{4(2)^4}{4} - \frac{3(2)^5}{5}\right]$$

$$= \frac{1}{2}\left[\frac{160}{3} - 16 - \frac{96}{5}\right]$$

$$= \frac{1}{2}\left[\frac{800-528}{15}\right]$$

$$= \frac{136}{15}$$

$$\therefore \int_0^2 \int_{x^2}^{2x} (2x + 3y)\, dydx = \frac{136}{15}$$

vii) $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^3\, dydx$

Solution:

If x is constant y varies from 0 to $\sqrt{a^2 - x^2}$. x varies from 0 to a

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^3\, dydx = \int_0^a \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{(\sqrt{a^2-x^2})^4}{4}\right] dx$$

$$= \int_0^a \frac{(a^2-x^2)^2}{4} dx$$

$$= \frac{1}{4}\int_0^a (a^4 + x^4 - 2a^2x^2) dx$$

$$= \frac{1}{4}\left[a^4x + \frac{x^5}{5} - 2a^2\frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{4}\left[\frac{15a^5+3a^5-10a^5}{15}\right]$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y^3\, dydx = \frac{2a^5}{15}.$$

viii) $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 dxdy$

Solution:

If y is constant x varies from $\sqrt{y}$ to $2-y$ and y is varies from 0 to 1

$$\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 dxdy = \int_0^1 \left[\frac{x^3}{3}\right]_{\sqrt{y}}^{2-y} dy$$

$$= \int_0^1 \left[\frac{(2-y)^3}{3} - \frac{(\sqrt{y})^3}{3}\right] dy$$

$$= \int_0^1 \frac{\left[2^3 - 3(2)^2 y + 3(2)(y)^2 - y^3 - (y)^{\frac{3}{2}}\right]}{3} dy$$

$$= \int_0^1 \frac{\left[8 - 12y + 6y^2 - y^3 - y^{\frac{3}{2}}\right]}{3} dy$$

$$= \frac{1}{3}\left[8y - 12.\frac{y^2}{2} + 6.\frac{y^3}{3} - \frac{y^4}{4} - \frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1$$

$$= \frac{1}{3}\left[8y - 6y^2 + 2y^3 - \frac{y^4}{4} - \frac{2y^{\frac{5}{2}}}{5}\right]_0^1$$

$$= \frac{1}{3}\left[8 - 6 + 2 - 1 - \frac{1}{4} - \frac{2}{5}\right]$$

$$= \frac{1}{3}\frac{[160 - 120 + 40 - 5 - 8]}{20}$$

$$= \frac{1}{60}[200 - 133]$$

$$= \frac{67}{60}$$

ix) $\int_0^{\pi/2} \int_0^{\infty} \frac{rdrd\theta}{(r^2+a^2)^2}$

Solution:

$$\int_0^{\pi/2} \int_0^{\infty} \frac{rdrd\theta}{(r^2+a^2)^2} = \int_0^{\pi/2} d\theta \int_0^{\infty} \frac{rdr}{(r^2+a^2)^2}$$

$$= [\theta]_0^{\pi/2} \int_0^{\infty} \frac{dt}{2t^2}$$

$$= \frac{\pi}{4}\left[-\frac{1}{t}\right]_{a^2}^{\infty} = \frac{\pi}{4}\left[-\frac{1}{\infty} + \frac{1}{a^2}\right]$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{rdrd\theta}{(r^2+a^2)^2}=\frac{\pi}{4a^2}$$

x) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(asin2\theta+bcos2\theta)d\theta d\phi$

Solution:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(asin2\theta+bcos2\theta)d\theta d\phi=\int_0^{\pi/2}(asin2\theta+bcos2\theta)d\theta\int_0^{\pi/2}d\phi$$

$$=\frac{\pi}{2}\left[-\frac{acos2\theta}{2}+\frac{bsin2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}\left[-\frac{acos\pi}{2}+\frac{bsin\pi}{2}-\left(\frac{acos0}{2}+\frac{bsin0}{2}\right)\right]$$

$$=\frac{\pi}{2}\left[\frac{a}{2}+\frac{a}{2}\right]$$

$$=\frac{\pi a}{2}$$

xi) $\int_{-\pi/2}^{\pi/2}\int_o^{2cos\theta}r^2drd\theta$

Solution:

$$\int_{-\pi/2}^{\pi/2}\int_o^{2cos\theta}r^2drd\theta=\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2cos\theta}d\theta$$

$$=\int_{-\pi/2}^{\pi/2}(2cos\theta)^3d\theta$$

$$=\frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos^3\theta$$

$$=\frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos^2\theta d(sin\theta)$$

$$=\frac{8}{3}\{[\cos^2\theta\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2sin^2\theta cos\theta d\theta$$

$$=\frac{8.2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta dsin\theta$$

$$= \frac{16}{3}[\frac{\sin^3\theta}{3}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{16}{9}[1—(-1)]$$

$$\int_{-\pi/2}^{\pi/2}\int_{o}^{2cos\theta} r^2 drd\theta = \frac{32}{9}$$

xii) $\int_0^{\pi}\int_0^{a(1+cos\theta)} r^2 sin\theta drd\theta$

Solution:

$$\int_0^{\pi}\int_0^{a(1+cos\theta)} r^2 sin\theta drd\theta = \int_0^{\pi}[\frac{r^3}{3}]_0^{a(1+cos\theta)} \sin\theta d\theta$$

$$= \frac{1}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi}(1+\cos\theta)^3 d(\cos\theta)$$

$$= -\frac{a^3}{3}[\frac{(1+cos\theta)^4}{4}]_0^{\pi}$$

$$= -\frac{a^3}{3}[(1+cos\pi)^4 - (1+cos0)^4]$$

$$= -\frac{a^3}{3}[0-(1)^4]$$

$$= -\frac{a^3}{12}[-16] = \frac{4a^3}{3}$$

$$\int_0^{\pi}\int_0^{a(1+cos\theta)} r^2 sin\theta drd\theta = \frac{4a^3}{3}$$

Problem 2:

Find the value of $\iint(a^2-x^2)dxdy$ taken over half the circle $x^2+y^2=a^2$ in the positive quadrant

Solution:

If we take x as constant, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a

$$\iint (a^2 - x^2)dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy(a^2 - x^2)$$

$$= \int_0^a (a^2 - x^2)[y]_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^a (a^2 - x^2)\sqrt{a^2 - x^2}\ \mathrm{dx}$$

$$= \int_0^a (a^2 - x^2)^{\frac{3}{2}} dx$$

Put $x = sin\theta$, $dx = acos\theta d\theta$

$x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} (a^2 - a^2\sin^2\theta)^{\frac{3}{2}} acos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^3(1 - \sin^2\theta)^{\frac{3}{2}} acos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^4(1 - \sin^2\theta)^{\frac{3}{2}} cos\theta d\theta$$

$$= a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= a^4 \int_0^{\frac{\pi}{2}} \cos^3\theta\, d(sin\theta)$$

$$= a^4\{[\cos^3\theta \sin\theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 3\sin^2\theta\cos^2\theta\, d\theta\}$$

$$= 3a^4 \int_0^{\frac{\pi}{2}} \left(\frac{sin2\theta}{2}\right)^2 d\theta$$

$$= \frac{3a^4}{4} \int_0^{\frac{\pi}{2}} \frac{1-cos4\theta}{2} d\theta$$

$$= \frac{3a^4}{8} [\theta - \frac{sin4\theta}{4}]_0^{\frac{\pi}{2}}$$

$$= \frac{3a^4}{8}\left(\frac{\pi}{2}\right) = \frac{3a^4\pi}{16}$$

Problem 3:

Change the order of integration in the following integrals.

i) $\iint \frac{x}{x^2+y^2} dydx$

ii) $\int_0^a \int_y^x \frac{x^2}{\sqrt{x^2+y^2}} dxdy$

iii) $\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dxdy$

iv) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) \, dxdy$

Solution:

i) $\iint \frac{x}{x^2+y^2} dydx$

Given x varies from y to a , y varies from 0 to a

By changing the order of integration, we first integrate with respect to y and with respect to x.

x varies from $y = 0$ to $y = x$

y varies from $x = 0$ to $x = a$

$$\int_{y=0}^{y=a} \int_{x=y}^{x=a} \frac{x}{x^2+y^2} dydx = \int_{x=0}^{x=a} \int_{y=0}^{y=x} \frac{x}{x^2+y^2} dydx$$

$$= \int_0^a \int_0^x \frac{x}{x^2+y^2} dydx$$

$$= \int_0^a \left[x.\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$since, \int \frac{1}{y^2+a^2} dy = \frac{1}{a}\tan^{-1}\frac{y}{a}$$

$$= \int_0^a \left[\tan^{-1}\frac{x}{x} - \tan^{-1}\frac{0}{x}\right] dx$$

$$= \int_0^a [\tan^{-1} 1 - \tan^{-1} 0] dx$$

$$= \int_0^a \frac{\pi}{4} dx$$

$$= \frac{\pi}{4} [x]_0^a$$

$$= \frac{\pi}{4} [a - 0]$$

$$= \frac{\pi}{4} (a)$$

$$\int_0^a \int_y^x \frac{x}{x^2 + y^2} dydx = \frac{\pi a}{4}$$

ii) $\int_0^a \int_y^x \frac{x^2}{\sqrt{x^2+y^2}} dxdy$

Solution:

Given x varies from $x = y$ to $x = a$

y varies from 0 to a

By changing the order of integration, we first integrate with respect to y and with respect to x

x varies from o to a

y varies from o to x

$$\int_{y=0}^{y=a} \int_{x=y}^{x=a} \frac{x^2}{\sqrt{x^2 + y^2}} dydx = \int_{x=0}^{x=a} \int_{y=0}^{y=x} \frac{x^2}{\sqrt{x^2 + y^2}} dydx$$

$$= \int_0^a \left[x^2 \log(y + \sqrt{x^2 + y^2})\right]_0^x dx$$

$$since, \int \frac{dx}{\sqrt{a^2 + x^2}} = \log(x + \sqrt{a^2 + x^2})$$

$$= \int_0^a \left[\left(x^2 \log\left(x + \sqrt{x^2 + x^2}\right)\right) - \left(x^2 \log\left(0 + \sqrt{x^2 + 0}\right)\right)\right] dx$$

$$= \int_0^a \left[\left(x^2 \log\left(x + \sqrt{2x^2}\right)\right) - \left(x^2 \log\left(\sqrt{x^2}\right)\right)\right] dx$$

$$= \int_0^a \left[\left(x^2 \log\left(x + \sqrt{2}x\right)\right) - \left(x^2 \log x\right)\right] dx$$

$$= \int_0^a \left[\left(x^2 \log x\left(1 + \sqrt{2}\right)\right) - \left(x^2 \log x\right)\right] dx$$

$$= \int_0^a \left[\left(x^2 \log x + x^2 \log\left(1 + \sqrt{2}\right) - x^2 \log x\right)\right] dx$$

$$= \int_0^a \left[x^2 \log\left(1 + \sqrt{2}\right)\right] dx$$

$$= \log\left(1 + \sqrt{2}\right) \int_0^a x^2 \, dx$$

$$= \log\left(1 + \sqrt{2}\right) \left[\frac{x^3}{3}\right]_0^a$$

$$= \log\left(1 + \sqrt{2}\right) \left(\frac{a^3}{3}\right)$$

$$= \left(\frac{a^3}{3}\right) \log\left(1 + \sqrt{2}\right)$$

iii) $\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dxdy$

Solution:

Given x varies from o to a

y varies from 0 to $2\sqrt{ax}$

By changing the order of integration, we first integrate with respect to x and with respect to y

x varies from $\frac{y^2}{4a}$ to a

y varies from o to 2a

$$\int_{x=0}^{x=a} \int_{y=0}^{y=2\sqrt{ax}} x^2 \, dxdy = \int_{y=0}^{y=2a} \int_{x=y^2/4a}^{x=a} x^2 \, dxdy$$

$$= \int_0^{2a} \left[\frac{x^3}{3}\right]_{y^2/4a}^{a} dy$$

$$= \frac{1}{3}\int_0^{2a} \left[a^3 - \left(\frac{y^2}{4a}\right)^3\right] dy$$

$$= \frac{1}{3}\int_0^{2a} \left[a^3 - \frac{y^6}{64a^3}\right] dy$$

$$= \frac{1}{3}\left[a^3 y - \frac{y^7}{64a^3.7}\right]_0^{2a}$$

$$= \frac{1}{3}\left[a^3(2a) - \frac{(2a)^7}{64a^3.7} - 0\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{128a^7}{64a^3.7}\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{2a^7}{7}\right]$$

$$= \frac{2a^4}{3}\left[1 - \frac{1}{7}\right]$$

$$= \frac{2a^4}{3}\left[\frac{6}{7}\right]$$

$$= \frac{4a^4}{7}$$

$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dxdy = \frac{4a^4}{7}$$

iv) $\int_0^3 \int_1^{\sqrt{4-y}} (x + y) \, dxdy$

Solution:

Given x varies from 1 to $\sqrt{4 - y}$

y varies from 0 to 3

$y = 0$	$x = 1$	$x = \sqrt{4-y}$
$x^2 = 4 - y$	$y = 4 - x^2$	$x^2 = 4 - y$
$x^2 = 4$	$y = 4 - 1 = 3$	$x^2 = -(y-4)$
$x = \pm 2$	$(1,3)$	$v(0,4)$

By changing the order of integration, we first integrate with respect to y and with respect to x

x varies from 1 to 2 , y varies from 0 to $4 - x^2$

$$\int_{y=0}^{y=3}\int_{x=1}^{x=\sqrt{4-y}} (x+y)\,dxdy = \int_{x=1}^{x=2}\int_{y=0}^{y=4-x^2} (x+y)\,dxdy$$

$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} - 0\right] dx$$

$$= \int_1^2 \left[4x - x^3 + \frac{16 + x^4 - 8x^2}{2}\right] dx$$

$$= \frac{1}{2}\int_1^2 [8x - 2x^2 + 16 + x^4 - 8x^2]\,dx$$

$$= \frac{1}{2}\left[\frac{8x^2}{2} - \frac{2x^4}{4} + 16x + \frac{x^5}{5} - \frac{8x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(4(4) - 8 + 16(2) + \frac{32}{5} - \frac{8(8)}{3}\right) - \left(4 - \frac{1}{4} + 16 + \frac{1}{5} - \frac{8}{3}\right)\right]$$

$$= \frac{1}{2}\left[16 - 8 + 32 + \frac{32}{5} - \frac{64}{3} - 4 + \frac{1}{4} - 16 - \frac{1}{5} + \frac{8}{3}\right]$$

$$= \frac{1}{2}\left[40 + \frac{32}{5} - \frac{64}{3} - 20 + \frac{1}{4} - \frac{1}{5} + \frac{8}{3}\right]$$

$$= \frac{1}{2}\left[20 + \frac{31}{5} - \frac{56}{3} + \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{1200 + 372 - 1120 + 15}{60}\right]$$

$$= \frac{1}{2}\left[\frac{467}{60}\right]$$

$$= \frac{467}{120}$$

$$= 3.89$$

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dxdy = 3.89$$

2.5. Double integral in polar co-ordinates:

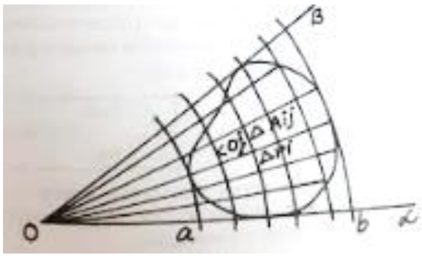

Figure 2.7

To evaluate the double integral in polar co-ordinates first integrate f(r, θ), r with respect to r keeping θ constant between the limits $r=f_1(\theta)$ and $r=f_2(\theta)$ and integrate the remaining expression with respect to θ between θ=α and θ=β,

$$\therefore \iint_R f(r,\theta) r\, dr\, d\theta = \int_\alpha^\beta \int_{r=f_1(\theta)}^{r=f_2(\theta)} r f(r,\theta) dr\, d\theta$$

Regarding dA as a rectangle its area will be product of a pair of adjacent sides say TU and UV

$$dA = (rd\theta)dr = r\, dr\, d\theta$$

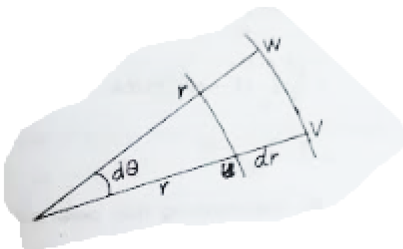

Figure 2.8

Hence the double integral in Cartesian form $\iint_R f(x,y)dx\,dy$ transforms into $\iint_R f(r\,cos\theta, r\,sin\theta)rdrd\theta$.

Example 1:

Evaluate $\iint r\sqrt{a^2-r^2}\,dr\,d\theta$ over the upper half of the circle r=a cos**θ**

Solution:

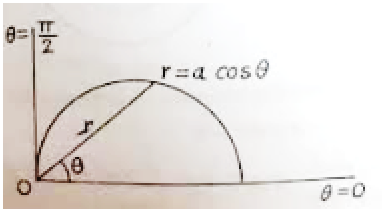

Figure 2.9

$\iint r\sqrt{a^2-r^2}\,dr\,d\theta = \int_0^{\pi/2}\int_0^{acos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^{acos\theta}(a^2-r^2)^{1/2}\,d(\frac{-r^2}{2})d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}[\frac{2}{3}(a^2-r^2)^{3/2}]_0^{acos\theta}\,d\theta$

$= -\frac{1}{3}\int_0^{\pi/2}[(a^2-a^2\cos^2\theta))^{3/2} - (a^2)^{3/2}]\,d\theta$

$= -\frac{1}{3}\int_0^{\pi/2}[a^2(1-\cos^2\theta))^{3/2} - (a)^3]\,d\theta$

$= -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$

$= \frac{a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$

$= \frac{a^3}{3}[\int_0^{\pi/2}d\theta - \int_0^{\pi/2}\sin^3\theta\,d\theta]$

$= \frac{a^3}{3}[[\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} sin^2\theta sin\theta d\theta]$

$= \frac{a^3}{3}[\frac{\pi}{2} - \int_0^{\pi/2} sin^2\theta d(-cos\theta)d\theta]$

$= \frac{a^3}{3}[\frac{\pi}{2} + [-\sin^2\theta\cos\theta + \frac{2\cos^3\theta}{3}]_0^{\frac{\pi}{2}}$

$= \frac{a^3}{3}[\frac{\pi}{2} - \frac{2}{3}]$

$= \frac{a^3}{3}[\frac{3\pi-4}{6}]$

$= \frac{a^3(3\pi-4)}{18}$

$\iint r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{a^3(3\pi-4)}{18}$

Example 2:

By transforming into polar co-ordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2}$ dxdy over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2<b^2$ (b>a)

Solution:

By transforming into polar co-ordinates, we get r=a & r=b.

$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dxdy = \int_0^{2\pi}\int_a^b \frac{r^2cos^2\theta\, r^2sin^2\theta}{r^2cos^2\theta + r^2sin^2\theta} r\,dr\,d\theta$

$= \int_0^{2\pi}\int_a^b \frac{r^4\, sin^2\theta cos^2\theta}{r^2} r\,dr\,d\theta$

$= \int_0^{2\pi}\int_a^b r^3\, sin^2\theta cos^2\theta r\,dr\,d\theta$

$= \int_0^{2\pi}[\frac{r^4}{4}]_a^b\, sin^2\theta\, cos^2\theta\,dr\,d\theta$

$= \frac{b^4-a^4}{4}\int_0^{2\pi}(\frac{sin2\theta}{2})^2\,dr\,d\theta$

$= \frac{b^4-a^4}{16}\int_0^{2\pi}(\frac{1-cos\theta}{2})^2\,d\theta$

$= \frac{b^4-a^4}{32}[2\pi]$

$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dxdy = \frac{\pi(b^4-a^4)}{16}$

Example 3:

By changing into polar co-ordinates evaluate the integral

$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dxdy.$

Solution:

The region bounded by x=0 & x=2a; y=0 & y=$\sqrt{2ax - x^2}$

(i.e.) The region of integration is the semicircle above the x-axis. Changing into polar, the region becomes r=2a cos θ & θ=0 to π/2

$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2 + y^2)dxdy = \int_0^{\pi/2}\int_0^{2acos\theta}(r^2cos^2\theta + r^2sin^2\theta)\text{rdrd}\theta$

$= \int_0^{\pi/2}\int_0^{2acos\theta} r^2(cos^2\theta + sin^2\theta)\text{rdrd}\theta$

$= \int_0^{\pi/2}\int_0^{2acos\theta} r^3\text{drd}\theta$

$= \int_0^{\pi/2}[\frac{r^4}{4}]_0^{2acos\theta}\, d\theta$

$= \int_0^{\pi/2}\frac{(2a\ cos\theta)}{4}\, d\theta$

$= 4a\int_0^{\pi/2} cos^4\theta d\theta$

$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2 + y^2)dxdy = \frac{3\pi a^4}{4}$

Exercises:

1. Evaluate $\iint r^3\ sin^2\theta\ dr\ d\theta$ over the area of the circle $r = a\cos\theta$
2. Evaluate $\iint r^2\ cos\ \theta\ dr\ d\theta$ over the loo of the lemniscate $r^2 = a^2\cos 2\theta$
3. Evaluate $\iint r^2\ sin\theta\ dr\ d\theta$ over the upper half of cardioid $r = a\,(1 + \cos\theta)$

UNIT-III

Triple integrals – applications of multiple integrals - volumes of solids of revolution - areas of curved surfaces – change of variables - Jacobian.

Chapter 3: Sections 3.1- 3.5.

3.1. Triple Integrals:

If $f(x, y, z)$ is continuous and a single valued function of x, y and z over the region of space R enclosed by the surface S.

$$\int_R f(x,y,z)dV = \int_{z_1}^{z_2} \int_{f_1(z)}^{f_2(z)} \int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)dxdydz.$$

The limits $z_1, z_2. f_1(z), f_2(z), \varphi_1(y,z), \varphi_2(y,z)$ can be determined from the equation of the surface S.

Note 1:

When integrating with respect to x in the above integral y, z are treated as constants and also when integrating with respect to y, z is treated as a constant.

Note 2:

When the integral is given $\iiint f(x,y,z)dx\,dy\,dz$ with limits, it is often these limits that show the order of integration. If the limits are not constants the integration should be in the order in which dx, dy, dz is given in the integral.

Example 1:

Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

Given the region of integration is the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$. In the region, z various from $o\ to \sqrt{a^2 - x^2 - y^2}$, y various form $0\ to \sqrt{a^2 - x^2}$, x various from $0\ to\ a$.

$$\iiint xyzdxdydz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyzdxdydz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2) dy\, dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2x\,y^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x}{2} - \frac{a^2x^3}{2} - \frac{a^2x^3}{2} + \frac{x^5}{2} - \frac{a^4x}{4} - \frac{x^5}{4} + \frac{2a^2x^3}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x}{4} - \frac{a^2x^3}{2} + \frac{x^5}{2}\right] dx$$

$$= \frac{1}{2}\left[\frac{a^4x}{4} - \frac{a^2x^3}{2} + \frac{x^5}{2}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^6}{8} - \frac{a^6}{8} + \frac{a^6}{24}\right]$$

$$= \frac{a^6}{48}$$

$$\iiint xyzdxdydz = \frac{a^6}{48}$$

3.2. Applications of Multiple Integrals

Find the area enclosed between x=a, x=b, by y=f(x), y=F(x).

Let P be (x, y) and Q be (x+$\Delta x, +\Delta y$). Hence the area of the rectangle PQRS is $\Delta x.\Delta y$. We may imagine the whole area divided into such elements and if we can sum them the required area is obtained.

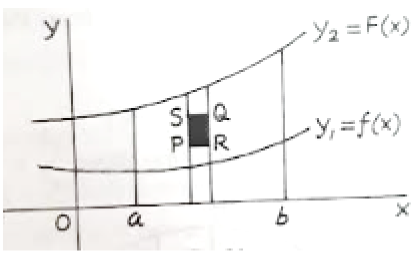

Figure 3.1

Regarding x and Δx as constants for the time being, the sum for the vertical strip y_1 *to* y_2 *is* $\underset{\Delta y \to 0}{\text{Lt}} \Delta x \sum_{y_1}^{y_2} \Delta y$.This can be written as $\Delta x \int_{y_1}^{y_2} dy$

Having found the area of the strip, we can add up all such strip between x=a and x=b, we get the required area as

$$\underset{\Delta x \to 0}{\text{Lt}} \sum_{a}^{b} \Delta x \int_{y_1}^{y_2} dy$$

(i.e.) $\int_a^b dx \int_{y_1}^{y_2} dy$

$$\int_a^b \int_{y_1}^{y_2} dy\, dx$$

In the same way the co-ordinates of the centre of gravity can be expressed as

$\bar{x} = \frac{\iint x\, dx\, dy}{\iint dx\, dy}$ $\bar{y} = \frac{\iint y\, dx\, dy}{\iint dx\, dy}$

The limits of integration to be taken to cover the given area.

Similarly the moment of inertia of an area about an axis through the origin perpendicular, to the xy-plane can be expressed as $\iint (x^2 + y^2)\, dx\, dy$, the limits being taken to cover the whole area since $x^2 + y^2$ is the square of the distance from the origin to the element of area.

Example 1:

Find the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

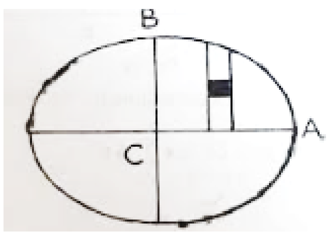

Figure 3.2

The area of the ellipse=4 (area in the first quadrant) =4 CAB

Keeping x constant for the time being, y varies from 0 to $\left\{\left(1-\frac{x^2}{a^2}\right)^2\right\}^{\frac{1}{2}}$ and then allowing x to vary. x varies from 0 to a

Hence the area of the ellipse $= \int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} dy\, dx$

Simplifying, we get the area as πab.

Example 2:

Find the centroid of the area enclosed by the parabola $y^2 = 4\,ax$ the axis of x and the latus rectum of the parabola.

Solution:

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid .

Then $\bar{x} = \frac{\int_0^a \int_0^{y_1} x\, dy\, dx}{\int_0^a \int_0^{y_1} dy\, dx}$

$$\bar{y} = \frac{\int_0^a \int_0^{y_1} y\, dy\, dx}{\int_0^a \int_0^{y_1} dy\, dx}$$

Where $y_1 = \sqrt{4ax}$

Simplifying, we get $\bar{x} = \left(\frac{3a}{5}, \frac{3a}{4}\right)$

Example 3:

A plane lamina of non-uniform density is in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If the density at any point (x,y) be K xy, where K is a constant, find the co-ordinates of the centroid of the lamina.

Solution:

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid.

Then $\bar{x} = \dfrac{\int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} K\, xy\, x\, dy\, dx}{\int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} K\, xy\, dy\, dx}$

$$\bar{y} = \frac{\int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} K\, xy\, y\, dy\, dx}{\int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} K\, xy\, dy\, dx}$$

Numerator of $\bar{x} = K \int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} x^2 y\ \ dy\, dx$

$$= \int_0^a K \left[\frac{y^2}{2}\right]_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} x^2\, dx$$

$$= \frac{K}{2} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) x^2\, dx$$

$$= K \frac{b^2}{2} \left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^a$$

$$= K \frac{b^2}{2} \left[\frac{a^3}{3} - \frac{a^5}{5}\right]$$

$$= K \frac{a^3 b^2}{15}$$

Denominator of $\bar{x} = K \int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} x\, y\ \ dy\, dx$

$$= K \int_0^a x \left[\frac{y^2}{2}\right]_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} dx$$

$$= \frac{K\, b^2}{2} \int_0^a x \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{K b^2}{2} \left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a$$

$$= \frac{Ka^2b^2}{8}$$

$\bar{x} = \frac{8a}{15}$, $\bar{y} = \frac{8b}{15}$

Example 4:

The density of the material of a right circular cylinder of radius a varies as the distance from the axis and as the distance from one end. find the radius of gyration about the axis.

Solution:

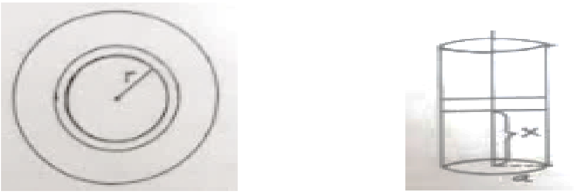

Figure 3.3

Take a section of the cylinder perpendicular to the axis at distances x and x+dx. It will be a circular strip of radius a. Take two concentric circles of radii r and $r + \Delta r$ on this circular strip and consider the solid between these two circles with thickness Δx as the element of volume dv.

$dv = 2\pi r\ \Delta r \Delta x$ and density $\rho = \lambda\ rx$

Then the mass of the element= $\lambda\ 2\pi r \Delta r \Delta x\ rx$

Hence the mass of the cylinder $M = \lambda \int_0^h \int_0^a 2\pi r^2\ x\ dr\ dx$

$$= \frac{\pi a^3 h^2}{3} \lambda$$

Moment of inertia of this element about the axis of the cylinder

$$= 2\pi\ \lambda r^2 x\ \Delta r \Delta x\ r^2$$

Hence the M.I of the cylinder about the axis

$$= \int_0^h \int_0^a 2\ \lambda r^4\ x\ dr\ dx$$

$$= \frac{\pi a^5 h^2}{5} \lambda$$

Let the radius of gyration be K

Then, $MK^2 = \frac{\pi a^5 h^2}{5}\lambda$

i.e. $\frac{\pi a^3 h^2}{3}\lambda K^2 = \frac{\pi a^5 h^2}{5}\lambda$

i.e. $K^2 = \frac{3a^2}{5}$

3.3. Volumes of Solids of revolution:

Let AB of a lane curve y=f(x). Let AB revolve about the x-axis. Let the co-ordinates of P,Q be (x, y), (x+$\Delta x, y + \Delta y$) respectively. Complete the rectangle PQRS.

The area of this rectangle=$\Delta x.\,\Delta y$. In making a complete revolution the area PQRS generates a solid whose volume is $\pi\{(y + \Delta y)^2 - y^2\}\Delta x$

i.e. $\pi\{2y\,\Delta y\,\Delta x + (\Delta y)^2\Delta x\}$

i.e., $2\pi\, y\,\Delta y\,\Delta x$ to the first order of infinitesimals.

Hence the total volume is

$$V = \underset{\Delta x \to 0}{\text{lt}} \sum_{x=a}^{x=b} \sum_{y=0}^{y=f(x)} 2\pi y \Delta y\, \Delta x$$

$$= 2\pi \int_a^b \int_0^{f(x)} y\, dy\, dx$$

Example 1:

Find the volume of a segment of height h of a sphere of radius a.

Solution:

The equation of the generating circle is $x^2 + y^2 = a^2$, the centre being the origin and the x-axis being perpendicular to the plane which cuts of the segment.

Volume of the segment= $2\pi \int_{a-h}^{a} \int_0^{\sqrt{a^2-x^2}} y\, dy\, dx$

$$= 2\pi \int_{a-h}^{a} \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \pi \int_{a-h}^{a} (a^2 - x^2)\, dx$$

$$= \frac{\pi h}{3} (3\, ah - h^2)$$

3.4. Areas of curved surfaces:

Let z=f(x, y) be the equation of the surface AB and suppose it is required to calculate the area of a region s' lying on the surface.

Project s'orthogonally on the XY plane and let S denote this region. Draw lines parallel to the x and y-axis and divide the area S into rectangles of areas $\Delta x.\Delta y$. Then the orthogonal projection of the element of area PQRT on the to the XOY plane is $\Delta x\ \Delta y$.

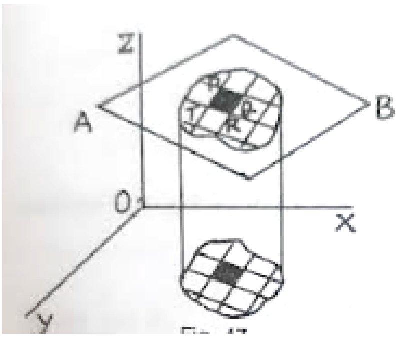

Figure 3.4

$\therefore\ \Delta x\ \Delta y = area\ of\ PQRT \cos\alpha$ where α is the angle between the XOY plane and the tangent plane at p, i.e., between the z-axis and the normal line perpendicular to the tangent plane at P.

The direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ and so in this particular case they are proportional to $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$

since $F(x, y, z) = f(x, y) - z = 0$

Hence $\cos\alpha = \frac{1}{\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}}}$

Area PQRT$= \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}}\ \Delta x\ \Delta y$

Hence $s' = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{\text{limit}} \sum\sum \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}} \Delta x\, \Delta y$

$$= \int\int_s \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}} dx\, dy$$

The limits of integration depending on the projection on the XOY plane of the region S'. We can easily show that it is also equal to

$$= \int\int_s \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2\right\}^{\frac{1}{2}} dz\, dx$$

Where S is the projection of S' in the XOZ plane or equal to

$$= \int\int_s \left\{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2\right\}^{\frac{1}{2}} dz\, dy$$

Where S is the projection of S' on the YOZ plane.

Example 1:

Find the area of the surface of the sphere of radius r.

Solution:

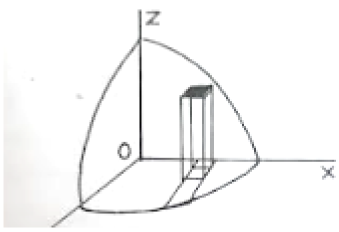

Figure 3.5

Taking the origin as the centre and radius r, the equation of the sphere is $x^2 + y^2 + z^2 = r^2$

Let us consider the surface of the sphere in the first octant. It will be $\frac{1}{8}$ of the surface of the sphere. The orthogonal projection of this surface area on the XOY plane is the quadrant of the circle $x^2 + y^2 = r^2$ in that plane.

Hence this surface area $\iint \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\}^{\frac{1}{2}} dy\, dx$

Taken over the area of the quarter of the circle $x^2 + y^2 = r^2$ on the positive quadrant

$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = -\frac{y}{z}$

$\therefore$ Surface area of the sphere $= 8 \iint \left\{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1\right\}^{\frac{1}{2}} dy\, dx$

$$= 8 \iint \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{z} dy\, dx$$

$$= 8 \iint \frac{r}{z} dy\, dx$$

$$= 8 \int_0^r \int_0^{\sqrt{r^2-x^2}} \frac{r\, dy\, dx}{\sqrt{r^2 - x^2 - y^2}}$$

$$= 8 \frac{\pi r^2}{2} = 4\pi r^2$$

Example 2:

Find the area of the surface of the sphere $x^2 + y^2 + z^2 = 9a^2$ cut-off by the cylinder

$x^2 + y^2 = 3ax.$

Solution:

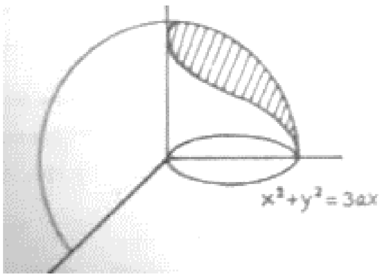

Figure 3.6

The Projection of the required area S on the xoy plane is the circle $x^2 + y^2 = 3ax$. On the sphere $z = \sqrt{9a^2 - x^2 - y^2}$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{9a^2-x^2-y^2}}$,

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{9a^2-x^2-y^2}}$$

$S = \int\int_R \left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{\frac{1}{2}} dx\, dy$ where R is the region enclosed by the circle

$x^2 + y^2 = 3ax$

$$= \int\int_R \left\{1 + \left(\frac{x^2}{9a^2-x^2-y^2}\right)^2 + \left(\frac{y^2}{9a^2-x^2-y^2}\right)^2\right\}^{\frac{1}{2}} dx\, dy$$

$$= \iint_R \frac{3a}{\sqrt{9a^2-x^2-y^2}}\, dx\, dy$$

$= 3a \iint_R \frac{r\, dr\, d\theta}{\sqrt{9a^2-r^2}}$ changing to polar.

The polar equation of the circle is r=3a cosθ. To cover the area of this circle r varies from 0 to 3a cosθ and θ from $-\frac{\pi}{2}$, $\frac{\pi}{2}$

$$\therefore S = 3a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{3a\cos\theta} \frac{r\, dr\, d\theta}{\sqrt{9a^2-r^2}}$$

$$= 9a^2(\pi - 2)$$

Example 3:

Find the portion of the cone $x^2 + y^2 = 4z^2$ lying above the xoy plane and inside the cylinder $x^2 + y^2 = 3y$

Solution:

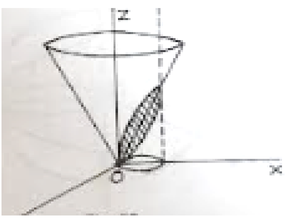

Figure 3.7

The projection of the required area on the xoy plane is the circle

$x^2 + y^2 = 3y$

On the cone z=$\frac{1}{2}\sqrt{x^2+y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{x}{\sqrt{x^2-y^2}} \qquad ,$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{y}{\sqrt{x^2-y^2}}$$

$$S = \int\int_R \left\{1 + \frac{1}{2}\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right\}^{\frac{1}{2}} dx\, dy$$

$= \frac{\sqrt{5}}{2}\int\int_R\ dx\, dy$, where R is the circle $x^2 + y^2 = 3y$

$$= \frac{\sqrt{5}}{2}\,\pi\left(\frac{3}{2}\right)^2$$

$$= \frac{9\sqrt{5}}{8}\,\pi$$

Example 4:

The centre of a sphere of radius r is on the surface of right cylinder, the radius of whose base is $\frac{r}{2}$. Find the area of the surface of the cylinder intercepted by the sphere.

Solution:

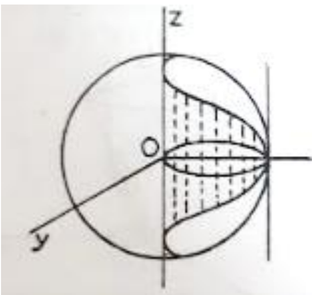

Figure 3.8

The equation of the sphere is $x^2 + y^2 + z^2 = r^2$ and the equation of the cylinder is

$(x - \frac{r}{2})^2 + y^2 = (\frac{r}{2})^2$

i.e., $x^2 + y^2 = rx$

Projecting on the zox plane, we have $S = \int\int_R \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2\right\}^{\frac{1}{2}} dz\, dx$

On the cylinder $y^2 = rx - x^2$

$\frac{\partial y}{\partial x} = \frac{r-2x}{2y}, \frac{\partial y}{\partial z} = 0$

The Projection of the area on the zox plane is the curve by eliminating y from $x^2 + y^2 = rx$

and $x^2 + y^2 + z^2 = r^2$

i.e., $z^2 + rx = r^2$

Hence the required area

$$= 2\int\int_R \left\{1 + \frac{(r-2x)^2}{4y^2}\right\}^{\frac{1}{2}} dz\, dx$$

$$= 2\iint \frac{\sqrt{4y^2 + 4x^2 - 4rx + r^2}}{2y}\, dz\, dx$$

$$= 2r\iint \frac{dz\, dx}{2y}$$

$$= r\iint \frac{dz\, dx}{\sqrt{rx - x^2}}$$

$$= r\int_0^r \int_{-\sqrt{r^2-rx}}^{\sqrt{r^2-rx}} \frac{dz\, dx}{\sqrt{rx - x^2}}$$

$$= 2r\int_0^r \frac{\sqrt{r^2-rx}}{\sqrt{rx-x^2}}\, \mathrm{dx}$$

$$= 2r\int_0^r \frac{\sqrt{r}}{\sqrt{x}}\, dx$$

$$= 4r^2.$$

3.5. Change of variables

Jacobian:

If $u = f(x,y), v = \varphi(x,y)$ be two continuous function of the independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y then $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

In case of three variables u, v, w which are functions of x, y, z the Jacobian of u, v, w with respect to x, y, z is defined as $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ as is denoted by $J\left(\frac{u,v,w}{x,y,z}\right)$ or $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Two important results regarding Jacobians:

Result:

If u, v are function of x, y and x, y are themselves function of ζ, η then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(\zeta,\eta)} = \frac{\partial(u,v)}{\partial(\zeta,\eta)}$

Now $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(\zeta,\eta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial \zeta} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} & \frac{\partial v}{\partial \eta} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \zeta} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \zeta} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial \zeta} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} & \frac{\partial v}{\partial \eta} \end{vmatrix} = \frac{\partial(u,v)}{\partial(\zeta,\eta)}$$

Result 2:

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$$

By 1,

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial(u,v)}{\partial(u,v)}$$

Now,

$$\frac{\partial(u,v)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial u} & \frac{\partial u}{\partial v}\\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}=1$$

Since u and v are independent variables

$$\frac{\partial u}{\partial v}=\frac{\partial v}{\partial u}=0$$

Hence

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}=1$$

Corollary:

For three variables

i) $\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{\partial(x,y,z)}{\partial(\zeta,\eta,\iota)}=\frac{\partial(u,v,w)}{\partial(\zeta,\eta,\iota)}$

ii) $\frac{\partial(u,v,w)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(u,v,w)}=1$

Example 1:

Given that $x+y=u, y=uv$ change the variables to u,v in the integral $\iint[xy(1-x-y)]^{\frac{1}{2}}dxdy$ taken over the area of the triangle with sides $x=0, y=0, x+y=1$ evaluate it.

Solution:

Given $x+y=u, y=uv$

$x=u-y$

$\Rightarrow x=u-uv \; and \; y=uv$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

Also, $x = 0 \Rightarrow u(1-v) = 0$

$u = 0\ and\ v = 1$ also $v = 0\ and\ u = 1$

$$\iint [xy(1-x-y)]^{\frac{1}{2}} dxdy = \int_0^1 \int_0^1 [u^2v(1-v)(1-u+uv-uv)]\text{^}(1/2)\ ududv$$

$$= \int_0^1 \int_0^1 [(u^2v - u^2v^2)(1-u)]^{\frac{1}{2}} ududθ$$

$$= \int_0^1 u^2(1-u)^{\frac{1}{2}} du \int_0^1 v^2(1-v)^{\frac{1}{2}} dv$$

$$= \frac{8\pi}{105.4} = \frac{2\pi}{105}$$

$$\iint [xy(1-x-y)]^{\frac{1}{2}} dxdy = \frac{2\pi}{105}$$

Example 2:

Evaluate $\iint (x-y)^4 e^{x+y} dxdy$ where R is the square with vertices $(1,0), (2,1), (1,2)$ & $(0,1)$

Solution:

Given the vertices of the square are $(1,0), (2,1), (1,2)$ & $(0,1)$

The region is bounded by $\frac{x-1}{2-1} = \frac{y-0}{1-0}$

$x - 1 = y \Longrightarrow x - y = 1$

$$\frac{x-2}{1-2} = \frac{y-1}{2-1}$$

$1(x-2) = -(y-1) \implies (x-2) = -y+1$

$x + y - 3 = 0$

$x + y = 3$

$\frac{x-1}{0-1} = \frac{y-2}{1-2}$

$-1(x-1) = -(y-2) \Longrightarrow\ -x+1 = -y+2$

$x - y + 1 = 0$

$x - y = -1$

$\frac{x-1}{0-1} = \frac{y-0}{1-0}$

$(x-1) = -(y) \Longrightarrow x - 1 = -y$

Let $u = x + y$ & $v = x - y$ & $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

Then $u = 1, u = 3, v = -1, v = 1$

$$\iint (x-y)^4 e^{x+y} dxdy = \int_1^{-1}\int_1^3 v^4 e^u (-\tfrac{1}{2}) dudv$$

$$= -\tfrac{1}{2}\int_1^{-1} v^4 dv \int_1^3 e^u\, du$$

$$= -\tfrac{1}{2}\left[\frac{v^5}{5}\right]_1^{-1} [e^u]_1^3 = -\tfrac{1}{2}\left[-\tfrac{1}{5} - \tfrac{1}{5}\right][e^3 - e^1]$$

$$= \frac{e^3 - e^1}{5}$$

Example 3:

Evaluate $\iint xydxdy$ where R is the region in the first quadrant bounded R by the hyperbolas $x^2 + y^2 = a^2$ & $x^2 - y^2 = b^2$ and the circles $x^2 + y^2 = c^2$ & $x^2 + y^2 = d^2$ $(0 < a < b < c < d)$

Solution:

Let $u = x^2 - y^2$ & $v = x^2 + y^2$

$u = a^2; u = b^2$ & $v = c^2; v = d^2$

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 4xy + 4xy = 8xy$

$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy}$

$\therefore \iint xydxdy = \int_{u=a^2}^{u=b^2}\int_{v=c^2}^{v=d^2} xy\frac{\partial(x,y)}{\partial(u,v)} dudv$

$$= \frac{1}{8}\int_{a^2}^{b^2}\int_{c^2}^{d^2} dudv$$

$$= \frac{1}{8}[u]_{a^2}^{b^2}[v]_{c^2}^{d^2}$$

$$\iint xydxdy = \frac{1}{8}(b^2 - a^2)(d^2 - c^2)$$

Example 4:

Find the area of the curvilinear quadrilateral bounded by the four parabolas $y^2 = ax, y^2 = bx, x^2 = cy, x^2 = dy.$

Solution:

Given $y^2 = ax, y^2 = bx, x^2 = cy, x^2 = dy.$

$$\Rightarrow \frac{y^2}{x} = a; \frac{y^2}{x} = b; \frac{x^2}{y} = c; \frac{x^2}{y} = d$$

Let $\frac{y^2}{x} = u$ & $\frac{x^2}{y} = v$

$$\Rightarrow u = a; u = b; v = c; v = d$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y^2}{x^2} & 2\frac{y}{x} \\ 2\frac{x}{y} & -\frac{x^2}{y^2} \end{vmatrix} = 1 - 4 = -3$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$$

$$\iint dxdy = \int_{u=a}^{b}\int_{v=c}^{d}\frac{\partial(x,y)}{\partial(u,v)}dudv$$

$$= \int_{u=a}^{b}\int_{v=c}^{d}(-\frac{1}{3})dudv$$

$$= -\frac{1}{3}[u]_a^b[v]_c^d = \frac{(b-a)(d-c)}{3}$$

UNIT-IV

Beta and Gamma functions – infinite integral – definitions – recurrence formula of Gamma functions – properties of Beta and Gamma functions - relation between Beta and Gamma functions - Applications.

Chapter 4: Sections 4.1- 4.6

Beta and Gamma functions

4.1. Definitions:

i) $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m > 0, n > 0$ is known as Beta function and

it is denoted by $\beta(m,n)$.

ii) $\int_0^\infty e^{-x}x^{n-1}dx$ for $n > 0$ is known as Gamma function and it is denoted by $\Gamma(n)$.

4.2.Convergence of $\Gamma(n)$

Theorem 1:

Prove that $\Gamma(n)$ convergence for $n > 0$.

Proof:

We know that, $\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx = \int_0^1 x^{n-1}\, e^{-x}\, dx + \int_1^2 x^{n-1}\, e^{-x} dx$

The first integral is $\lim_{\varepsilon \to 0} \int_{\varepsilon_t}^1 x^{n-1}\, e^{-x}\, dx$ if this limit exists.

when x is small, the integral behaves like x^{n-1} and the limit exists if $n > 0$. The second integral certainly exists for $e^x > \frac{x^r}{r!}$ (r being any positive integer) $> \frac{x^{n+1}}{r!}$

so long as $r > n + 1$.

Hence $x^{n-1}e^{-x} < \frac{r!}{x^2}$

$\therefore \int_0^\infty e^{-x}x^{n-1}dx$ does not excess a constant multiple of $\int_1^\infty \frac{dx}{x^2}$ which converges.

$\therefore \Gamma(n)$ converges for $n > 0$.

Corollary:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$\beta(m,n)$ exists if $m > 0, n > 0$

4.3. Recurrence formula of gamma functions:

$\Gamma(n+1) = n\Gamma(n)$

Prove that: $\Gamma(n+1) = n\Gamma(n)$ if $n > 0$

We know that, $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{n+1-1}dx$$

$$= \int_0^\infty e^{-x}x^{n}dx;\ n \geq 0$$

$$= -x^n e^{-x} \int_0^\infty -ndx \int_0^\infty e^{-x}x^{n-1}dx$$

$$= \lim_{a\to\infty}(-x^n e^{-x})_0^a + n\int_0^\infty e^{-x}x^{n}dx$$

$$= n\int_0^\infty e^{-x}x^{n-1}dx$$

$\therefore \Gamma(n+1) = n\Gamma(n)$

Note: This recurrence formula is true only when $n > 0$

Corollary 1:

$\Gamma(n+1) = n!$

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}dx$$

$$= \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= \lim_{a\to\infty}(-e^{-x})_0^\infty = -e^{-\infty} + e^0$$

$$\Gamma(1) = e^0 = 1$$

2) $\Gamma(n+a) = (n+a-1)(n+a-2)\ldots a\Gamma(a)$

4.4. Properties of Beta Function:

i) $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Putting $x = 1-y$, we have

$$\beta(m,n) = \int_0^1 y^{n-1}(1-y)^{m-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= \beta(n,m)$$

[This is merely a property of integrals, viz., $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

ii) $\beta(m,n) = \int_0^a \frac{y^{m-1}}{(1+y)^{m+n}}dy$

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad \ldots\ldots\ldots (1)$$

Put $x = \frac{y}{1+y}$

$$dx = \frac{(1+y)dy - ydy}{(1+y)^2}$$

$$= \frac{dy + ydy - ydy}{(1+y)^2} = \frac{dy}{(1+y)^2}$$

$$(1) \Longrightarrow \beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}}\left(\frac{1+y-y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}}\left(\frac{1}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

iii) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ .... (1)}$$

Put $x = \sin^2\theta$

$$dx = 2\sin\theta\cos\theta$$

$$(1) \Longrightarrow \beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\cos\theta\sin\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2+1}\theta\cos^{2n-2+1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

Which can be written as

$$\beta(m,n) = \frac{1}{2}\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

4.5. Relation between Beta and Gamma functions:

Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof:

We have $\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$ and $\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x+y)^2}x^{2m-1}y^{2n-1}dx\,dy$$

Put $x = rcos\theta$, $y = rsin\theta$ and $x^2 + y^2 = r^2$, $dy\,dx = r\,dr\,d\theta$

$$J = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\mathrm{r}\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} (\mathrm{r}\cos\theta)^{2m-1} (\mathrm{r}\sin\theta)^{2n-1} r d\theta\, dr$$

$$= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-2} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} r d\theta\, dr$$

$$= 4\int_0^\infty e^{-r^2} (r^2)^{m+n-1} \frac{1}{2} d(r^2) \int_0^{\frac{\pi}{2}} (\sin\theta)^{2n-1} d\theta$$

$$= 4.\frac{1}{2}\,\Gamma(m+n).\frac{1}{2}\,\beta(n,m)$$

$$= \Gamma(m+n)\beta(m,n)$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\beta(m,n) \qquad [\,\beta(m,n) = \beta(n,m)]$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Corollary 1: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

We have, $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin x)^{2m-1} (\cos x)^{2n-1}\, dx$

Put $m = \frac{1}{2}$ and $n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} (\sin x)^{2\left(\frac{1}{2}\right)-1} (\cos x)^{2\left(\frac{1}{2}\right)-1}\, dx$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} dx = 2[x]_0^{\frac{\pi}{2}} \qquad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$$\left(\Gamma(\frac{1}{2})\right)^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Corollary 2:

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta = \frac{1}{2}\beta(m,n)$$

Putting 2m=p and 2n=q

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}(\cos\theta)^{q-1}\,d\theta = \frac{1}{2}\beta\left(\frac{p}{2},\frac{q}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}{\Gamma(\frac{p}{2}+\frac{q}{2})} \quad \text{………. (1)}$$

If we put q=1 in (1), we get

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}\,d\theta = \frac{1}{2}\frac{\Gamma(\frac{p}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p}{2}+\frac{1}{2})} \quad \text{……….. (2)}$$

If we put p=q in (1), we get

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}(\cos\theta)^{p-1}\,d\theta = \frac{1}{2}\frac{\left(\Gamma(\frac{p}{2})\right)^2}{\Gamma(p)}$$

$$\frac{1}{2^{p-1}}\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}\,2\theta d\theta = \frac{\Gamma\left((\frac{p}{2})\right)^2}{2\Gamma(p)}$$

Putting $2\theta = \varphi$, we get

$$\frac{1}{2^{p-1}}\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}\,\varphi d\varphi = \frac{\left(\Gamma(\frac{p}{2})\right)^2}{\Gamma(p)}$$

$$\frac{2}{2^{p-1}}\int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}\,\varphi d\varphi = \frac{\left(\Gamma(\frac{p}{2})\right)^2}{\Gamma(p)}$$

Using equation (2), we get

$$\frac{\sqrt{\pi}}{2^{p-1}\Gamma(\frac{p+1}{2})} = \frac{\Gamma(\frac{p}{2})}{\Gamma(p)}$$

$$\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\,\Gamma(p) \quad \text{………. (3)}$$

Putting p=2n, we have

$$\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n) \quad \text{………… (4)}$$

Put $n = \frac{1}{4}$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}{2^{-\frac{1}{2}}} = \sqrt{2}\,\pi.$$

Example 1:

Evaluate $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx$

Solution:

Put $\log\frac{1}{x} = t$

$\Longrightarrow e^t = \frac{1}{x}$

$\Longrightarrow x = e^{-t}$

$dx = -e^{-t}dt$

When $x = 0\,,\; t = \log\infty = \infty$

$x = 1\,,\; t = \log 1 = 0$

$$\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \int_\infty^0 (\,e^{-t})^m t^n(-e^{-t}dt)$$

$$= -\int_\infty^0 e^{-mt-t}t^n dt$$

$$= -\int_\infty^0 e^{-(m+1)t}t^n dt$$

$$= -\int_0^\infty e^{-(m+1)t}t^n dt$$

Put $(m+1)t = y \quad \Longrightarrow \quad t = \frac{y}{1+m} \quad \Longrightarrow \quad dt = \frac{dy}{1+m}$

When, $t = 0, y = 0$

$t = \infty, y = \infty$

$$= \int_0^\infty \left(\frac{y}{1+m}\right)^n e^{-y}\,\frac{dy}{1+m}$$

$$= \int_0^\infty \frac{y^n}{(1+m)^n} e^{-y} \frac{dy}{1+m}$$

$$= \int_0^\infty \frac{y^n}{(1+m)^{n+1}} e^{-y} dy$$

$$= \frac{1}{(1+m)^{n+1}} \int_0^\infty y^n e^{-y} dy$$

$$\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \frac{1}{(1+m)^{n+1}} \Gamma(n+1) .$$

Example 2:

Evaluate $\int_0^\infty e^{-x^2} dx$

Solution:

$I = \int_0^\infty e^{-x^2} dx$

Put $x^2 = t \implies 2xdx = dt$

$\implies x = \sqrt{t} \implies dx = \frac{dt}{2\sqrt{t}}$

When $x = 0 , \quad t = 0$

$x = \infty , \quad t = \infty$

$I = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$

$= \int_0^\infty t^{-\frac{1}{2}} e^{-t} \frac{dt}{2}$

$= \frac{1}{2} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$

$I = \frac{1}{2} \Gamma\left(-\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$

$I = \frac{\sqrt{\pi}}{2}$

Example 3:

Evaluate i) $\int_0^1 x^7 (1-x)^8 dx$

ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta$

iii) $\int_0^{\frac{\pi}{2}} \sin^{10}\theta \, d\theta$

iv) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} \, d\theta$

Solution:

$\int_0^1 x^7 (1-x)^8 dx = \beta(7+1, 8+1)$

$= \beta(8,9)$

$= \frac{\Gamma(8)\Gamma(9)}{\Gamma(8+9)}$

$= \frac{7!8!}{16!}$

ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta$

$\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\beta(4,3)$

$= \frac{1}{2}\frac{\Gamma(4)\Gamma(3)}{\Gamma(4+3)}$

$= \frac{1}{2}\frac{3!2!}{6!}$

$= \frac{1}{2\times 60}$

$= \frac{1}{120}$

$\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{120}$

iii) $\int_0^{\frac{\pi}{2}} \sin^{10}\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{10}\theta \cos^0\theta \, d\theta$

$= \frac{1}{2}\beta(\frac{11}{2}, \frac{1}{2})$

$= \frac{1}{2}\frac{\Gamma(\frac{11}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{11}{2}+\frac{1}{2})}$

$$=\frac{1}{2}\frac{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(6)}$$

$$=\frac{9\times7\times5\times3\times1\times\sqrt{\pi}\times\sqrt{\pi}}{2^{6}5!}$$

$$=\frac{63\pi}{512}$$

$$\int_0^{\frac{\pi}{2}}\sin^{10}\theta\,d\theta=\frac{63\pi}{512}$$

iv) $\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{1}{2}}\,d\theta\,(\cos\theta)^{\frac{1}{2}}d\theta$$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta)^{2(\frac{3}{4}-1)}\,d\theta\,(\cos\theta)^{2(\frac{1}{4}-1)}d\theta$$

$$=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})$$

$$=\frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}+\frac{1}{4})}$$

$$=\frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(11)}$$

$$=\frac{1}{2}\,\Gamma(1-\frac{1}{4})\Gamma(\frac{1}{4})$$

$$=\frac{\pi}{2\sin\frac{\pi}{4}}$$

$$=\frac{\pi}{\sqrt{2}}$$

Example 4:

Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of Gamma functions and evaluate the integral $\int_0^1 x^5(1-x^3)^{10}dx$

Solution:

Put $x^n=y \implies x=y^{\frac{1}{n}}$

$$nx^{n-1}dx=dy$$

Where $x=0$ to $y=0$

$$\int_0^1(y^{\frac{1}{n}})^m(1-y)^p\,\frac{dy}{n(y^{\frac{1}{n}})^{n-1}}=\frac{1}{n}\int_0^1\frac{y^{\frac{m}{n}}(1-y)^p}{y^{\left(\frac{n-1}{n}\right)}}dy$$

$$= \frac{1}{n}\int_0^1 y^m - \left(\frac{n-1}{n}\right)(1-y)^p dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}}(1-y)^p dy$$

$$= \frac{1}{n}\beta\left(\frac{m-n+1}{n}, p+1\right)$$

$$= \frac{1}{n}\frac{\Gamma(\frac{m-n+1+n}{n})\Gamma(p+1)}{\Gamma(\frac{m-n+1+n}{n}+p+1)}$$

$$= \frac{1}{n}\frac{\Gamma(\frac{m+1}{n})\Gamma(p+1)}{\Gamma(\frac{m+1}{n}+p+1)}$$

$$\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{3}\frac{\Gamma(\frac{5+1}{3})\Gamma(10+1)}{\Gamma(\frac{5+1}{3}+10+1)}$$

$$= \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)}$$

$$= \frac{1}{3}\frac{1\times\Gamma(11)}{12\times 11\Gamma(11)}$$

$$= \frac{1}{396}$$

Example 5:

Prove that $\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(acos^2\theta + bsin^2\theta)^{m+n}} d\theta = \frac{\beta(m,n)}{2a^m b^n}$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(acos^2\theta + bsin^2\theta)^{m+n}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{cos^{2m+2n}\theta(a + btan^2\theta)^{m+n}} d\theta$$

$$\tan\theta = t \qquad \theta = 0 \Longrightarrow t = 0$$

$$sec^2\theta d\theta = dt \qquad \theta = \frac{\pi}{2} \Longrightarrow t = \infty \qquad \frac{d\theta}{cos^2\theta} = dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\cos\theta}\right)^{2n-1} \frac{d\theta}{cos^2\theta(a + btan^2\theta)^{m+n}}$$

$$= \int_0^{\infty} (t)^{2n-1} \frac{dt}{(a+bt^2)^{m+n}}$$

$\sqrt{b}t = \sqrt{ay}$

$\sqrt{b}dt = \frac{1}{2}(ay)^{-\frac{1}{2}}ady$

$= \frac{ady}{2\sqrt{ay}}$

$$\int_0^{\infty}(t)^{2n-1}\frac{dt}{(a+\sqrt{b}t)^2)^{m+n}} = \int_0^{\infty}\frac{\left(\sqrt{\frac{ay}{b}}\right)^{2n-1}}{(a+\sqrt{ay})^2)^{m+n}}\frac{ady}{2\sqrt{b}\sqrt{ay}}$$

$$= \int_0^{\infty}\frac{\left(\left(\sqrt{\frac{ay}{b}}\right)^2\right)^n}{a^{m+n}(1+y)^{m+n}}\frac{ady}{2\sqrt{b}\sqrt{ay}\frac{\sqrt{ay}}{6}}$$

$$= \frac{1}{2b^n a^{m+n}} = \int_0^{\infty}(y)^{n-1}\frac{dy}{(1+y)^{m+n}}$$

$$= \frac{\beta(m,n)}{2a^m b^n}$$

Exercises Problem:

Problem 1:

Show that $\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5\ldots\ldots\ldots(2n-1)}{2^n}\sqrt{\pi}$

Solution:

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)\ldots5\times3\times1}{2^n}\times\sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5\ldots\ldots\ldots(2n-1)}{2^n}\sqrt{\pi}$$

Problem 2:

Show that $\frac{1.3.5\ldots\ldots\ldots(2n-1)}{2.4.6\ldots..2n}=\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$

Solution:

$$\Gamma\left(n+\frac{1}{2}\right)=\Gamma\left(n+\frac{1}{2}-1\right)\Gamma\left(n+\frac{1}{2}-2\right)\ldots\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{(2n-1)(2n-3)\ldots5\times3\times1}{2^n}\times\sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\times3\times5\ldots\ldots\ldots\times(2n-1)}{2^n}\sqrt{\pi}$$

$$\Gamma(n+1)=(n+1-1)(n+1-2)\ldots2.1$$

$$\Gamma(n+1)=(n)(n-1)\ldots2.1$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}=\frac{1.3.5\ldots\ldots\ldots\ldots(2n-1)\sqrt{\pi}}{2^n[(n)(n-1)\ldots2.1]}$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}=\frac{1.3.5\ldots\ldots\ldots\ldots(2n-1)\sqrt{\pi}}{2.4.6\ldots2n}$$

Problem 3:

Show that $\frac{\beta(p,q+1)}{q}=\frac{\beta(q+1,p)}{p}=\frac{\beta(p,q)}{p+q}$

Solution:

$$\frac{\beta(p,q+1)}{q}=\frac{1}{q}\frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$=\frac{1}{q}\frac{\Gamma(p).q\Gamma(q)}{(p+q).\Gamma(p+q)}$$

$$\frac{\beta(p,q+1)}{q}=\frac{\Gamma(p).\Gamma(q)}{(p+q).\Gamma(p+q)} \qquad \ldots\ldots\ldots\ (1)$$

$$\frac{\beta(q+1,p)}{p}=\frac{1}{p}\frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)}$$

$$=\frac{1}{p}\frac{\mathrm{p}.\Gamma(p).\Gamma(q)}{(p+q).\Gamma(p+q)}$$

$\frac{\beta(q+1,p)}{p} = \frac{\Gamma(p).\Gamma(q)}{(p+q).\Gamma(p+q)}$ .... (2)

$\frac{\beta(p,q)}{p+q} = \frac{\Gamma(p).\Gamma(q)}{(p+q).\Gamma(p+q)}$ .... (3)

From (1), (2) & (3) equation $\frac{\beta(p,q+1)}{q} = \frac{\beta(q+1,p)}{p} = \frac{\beta(p,q)}{p+q}$

Problem 4:

Evaluate the integrals $\int_0^\infty e^{-x^3} dx$

Solution:

Put $x^3 = y$

$3x^2 dx = dy$

$dx = \frac{dy}{3x^2} = \frac{dy}{3\left(y^{\frac{1}{3}}\right)^2} = \frac{dy}{3y^{\frac{2}{3}}}$

$$dx = \frac{dy}{3y^{\frac{2}{3}}}$$

When $x = 0$, $y = 0$

$x = \infty$, $y = \infty$

$\int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-y} \frac{dy}{3y^{\frac{2}{3}}} = \frac{1}{3}\int_0^\infty y^{-\frac{2}{3}} e^{-y} dy$

$= \frac{1}{3}\Gamma\left(-\frac{2}{3}+1\right)$

$= \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$

$= \Gamma\left(\frac{4}{3}\right)$

$\int_0^\infty e^{-x^3} dx = \Gamma\left(\frac{4}{3}\right)$

Problem 5:

Evaluate the integral $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

Solution:

We have, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x^3 = y$

$3x^2dx = dy$

$dx = \frac{dy}{3x^2} = \frac{dy}{3\left(y^{\frac{1}{3}}\right)^2} = \frac{dy}{3y^{\frac{2}{3}}}$

$dx = \frac{dy}{3y^{\frac{2}{3}}}$

When $x = 0, y = 0$

$x = 1, y = 1$

$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 (1-x^3)^{-\frac{1}{2}}dx$

$= \int_0^1 (1-y)^{-\frac{1}{2}} \frac{dy}{3y^{\frac{2}{3}}}$

$= \frac{1}{3}\int_0^1 (1-y)^{-\frac{2}{3}} y^{-\frac{2}{3}} dy$

$= \frac{1}{3}\int_0^1 (1-y)^{\frac{1}{2}-1} y^{-1+\frac{1}{3}} dy$

$= \frac{1}{3}\left[\beta\left(\frac{1}{3},\frac{1}{2}\right)\right]$

$= \frac{1}{3}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3}+\frac{1}{2})}$

$= \frac{1}{3}\frac{\Gamma(\frac{1}{3})\sqrt{\pi}}{\Gamma(\frac{5}{6})}$

$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\pi}}{3}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})}.$

Problem 6:

Show that $\int_0^{\frac{\pi}{2}} \sqrt{sin\theta\, d\theta} \int_0^{\pi/2} \frac{1}{\sqrt{sin\theta}} d\theta = \pi$

Solution:

We know that, $\int_0^{\frac{\pi}{2}}(sin\theta)^{2m-1}\ (cos\theta)^{2n-1}d\theta = \frac{1}{2}\beta(m,n)$

Now, $\int_0^{\pi/2}\sqrt{sin\theta}d\theta = \int_0^{\pi/2}(sin\theta)^{\frac{1}{2}}d\theta$

$$= \int_0^{\frac{\pi}{2}}(sin\theta)^{2\left(\frac{3}{4}\right)-1}\ (cos\theta)^{2\left(\frac{1}{2}\right)-1}d\theta$$

$$= \frac{1}{2}\beta(\frac{3}{4},\frac{1}{2})$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\frac{\left(\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}}\sqrt{sin\theta}d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)} \qquad \ldots\ldots.(1)$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{sin\theta}}d\theta = \int_0^{\frac{\pi}{2}}(sin\theta)^{-\frac{1}{2}}d\theta$$

$$= \int_0^{\frac{\pi}{2}}(sin\theta)^{2\left(\frac{1}{4}\right)-1}\ (cos\theta)^{2\left(\frac{1}{2}\right)-1}d\theta$$

$$= \frac{1}{2}\beta(\frac{1}{4},\frac{1}{2})$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\frac{\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \ldots\ldots\ldots.. (2)$$

From (1) & (2) equation,

L.H.S $\int_0^{\frac{\pi}{2}}\sqrt{sin\theta\ d\theta}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{sin\theta}}d\theta = \frac{\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)} * \frac{\frac{1}{2}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\right)}{\Gamma\left(\frac{3}{4}\right)}$

$$= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\pi}\sqrt{\pi}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{sin\theta\, d\theta} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{sin\theta}} d\theta = \pi$$

Problem 7:

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} \cdot \int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \frac{\pi}{4}$$

Solution:

Let $x^2 = sin\theta\,,\; y^2 = tan\theta$

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} \frac{sin\theta cos\theta}{(1-\sin^2\theta)^{\frac{1}{2}}} \cdot \frac{1}{2\sin^{\frac{1}{2}}\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{sin\theta cos\theta}{cos\theta . 2(sin\theta)^{\frac{1}{2}}\theta}\, d\theta$$

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} = \frac{1}{2} \int_0^{\frac{\pi}{2}} (sin\theta)^{\frac{1}{2}}\, d\theta$$

[when $x = 0 \Rightarrow \theta = 0\;,\; x = 1 \Rightarrow \theta = \pi/2$]

$$= \frac{1}{2} \int_0^{\pi} (sin\theta)^{2\left(\frac{3}{4}\right)-1} . (cos\theta)^{2\left(\frac{1}{2}\right)-1} d\theta$$

$$= \frac{1}{2*2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} = \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \quad \ldots\ldots\ldots(1)$$

$$\int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}}$$

Put $x = y \Rightarrow dx = dy$

When $x = 0 \Rightarrow y = 0\; and\; x = 1 \Rightarrow y = 1$

$$\int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \int_o^1 \frac{dx}{(1-y^4)^{\frac{1}{2}}} = \int_o^1 (1-y^4)^{-\frac{1}{2}} dy$$

Put $y^2 = tan\theta$, $2y\,dy = \sec^2\theta\, d\theta$,

When $y = 0 \Rightarrow \theta = 0$ and $y = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \int_0^{\pi/4} (1-\tan^2\theta)^{-\frac{1}{2}} \frac{\sec^2\theta}{2\tan^{\frac{1}{2}}\theta} d\theta$$

$$= \int_0^{\pi/4} (\sec^2\theta)^{-\frac{1}{2}} \frac{1}{\cos^2\theta} \frac{\cos^{\frac{1}{2}}\theta}{\sin^{\frac{1}{2}}\theta} d\theta$$

$$= \int_0^{\pi/4} cos\theta \cos^{-\frac{3}{2}}\theta \sin^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2} . \frac{1}{2} \int_0^{\pi/4} \cos^{-\frac{1}{2}}\theta \sin^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/4} \sin^{2\left(\frac{1}{4}\right)-1}\theta \cos^{2\left(\frac{1}{4}\right)-1}\theta\, d\theta$$

$$= \frac{1}{4} . \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \quad \ldots\ldots. (2)$$

From (1) & (2) equation

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} \int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} . \frac{1}{8} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{1}{8} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right)$$

$$= \frac{\sqrt{2}\pi}{8}$$

$$\int_o^1 \frac{x^2 dx}{(1-x)^{\frac{1}{2}}} \int_o^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \frac{\pi}{4\sqrt{2}}$$

Problem 8:

Deduce from $\beta(m,n) = \int_0^\infty \frac{x^{m+n}}{(1+x)^{m+n}}$

Solution:

Let $x = \frac{y}{(1+y)} \Rightarrow y = \frac{x}{1-x}$

$$dx = d\left(\frac{y}{(1+y)}\right) = \frac{(1+y).1 - y.1}{(1+y)^2}dy$$

$$dx = \frac{dy}{(1+y)^2}$$

When $x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = \infty$

$\beta(m,n) = \int_0^1 x^{(m-1)}(1-x)^{n-1}dx$

$$= \int_0^\infty \left(\frac{y}{1+y)}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Exercises:

1. Show that $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} = \frac{1}{2}\beta\left(n+\frac{1}{2}, \frac{1}{2}\right)$
2. Evaluate the integrals $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$
3. Evaluate the integral $\int_0^\infty e^{-x^4}dx$
4. Show that $\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{\pi}{2}\sec\frac{\pi\alpha}{2}$
5. Show that $\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx = \frac{5\pi\sqrt{2}}{128}$

4.6. Applications of Gamma Functions to Multiple Integrals:

Example 1:

Evaluate the integral $\iint x^p\, y^q\, dy\, dx$ over the triangle x>0, y>0, $x + y \leq 1$ in terms of Gamma functions.

$\iint x^p\, y^q\, dy\, dx$ over the triangle OAB, where OA and OB are the intercepts on the axes.

$$= \int_0^1 x^p \left[\frac{y^{q+1}}{q+1}\right]_0^{1-x} dx$$

$$= \frac{1}{q+1}\int_0^1 x^p\,(1-x)^{q+1}\; dx$$

$$= \frac{1}{q+1}\,\beta(p+1, q+2)$$

$$= \frac{1}{q+1}\,\frac{\Gamma\,(p+1)\Gamma(q+2)}{\Gamma(p+q+3)}$$

$$= \frac{\Gamma\,(p+1)\Gamma(q+1)}{\Gamma(p+q+3)}$$

Example 2:

Evaluate the integral $\iint x^p\, y^q\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ in terms of Gamma functions. Deduce (i) the area of the circle, and (ii) the co-ordinate of the centroid of a quadrant of the circle.

Solution:

The quadrant of the circle is given by the equations

$$x \geq 0\,,\, y \geq 0,\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 \leq 1$$

Hence put $\left(\frac{x}{a}\right) = X^{\frac{1}{2}}\;\left(\frac{y}{a}\right) = Y^{\frac{1}{2}}$

Then the required integral becomes

$$\iint \left(a\,X^{\frac{1}{2}}\right)^p \left(a\,Y^{\frac{1}{2}}\right)^q \frac{1}{2}\,aX^{\frac{-1}{2}} \,.\, \frac{1}{2} aY^{\frac{-1}{2}}\ \text{dx dy}$$

i.e., $\frac{a^{p+q+2}}{4} \iint X^{\frac{p-1}{2}} Y^{\frac{q-2}{2}} dX\, dY$

over the region $x \geq 0$, $y \geq 0$, $X + Y \leq 1$

Hence its value is

$$\frac{a^{p+q+2}}{4} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

(i) Area of the circle is 4 $\iint dx\, dy$ over the region

$x \geq 0$, $y \geq 0$,$x^2 + y^2 \leq a^2$

In this case p=0, q=0.

Area of the circle= $4.\frac{a^2}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$

$= \pi a^2.$

(ii) Let $(\overline{x}\ ,\overline{y}\ )$ be the co-ordinates of the centroid of the quadrant of the circle.

$\overline{x} = \frac{\iint xy\, dy\, dx}{\iint dy\, dx}$, both the integrals being taken over the region $x \geq 0$, $y \geq 0$,$x^2 + y^2 \leq a^2$

In the integral $\iint x^p\, y^q\, dx\, dy$ if we put =1, q=0, we get the numerator.

Numerator= $\frac{a^3}{4} \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}$

$= \frac{a^3}{4} \frac{1\sqrt{\pi}}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}$

$= \frac{a^3}{3}$

$$\bar{x} = \frac{\frac{a^3}{3}}{\frac{1}{4}\pi a^2} = \frac{4a}{3\pi}$$

Similarly $\overline{y} = \frac{4a}{3\pi}$.

Hence the centroid is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$

Example 3:

Evaluate in terms of Gamma functions the integral $\iint x^p\, y^q\, z^r\, dx\, dy\, dz$ taken over the volume of the tetrahedron given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $X + Y + Z \ \leq 1$.

Solution:

The given integral= $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^p y^q\, z^r\, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} x^p y^q \left[\frac{z^{r+1}}{r+1}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{r+1} \int_0^1 \int_0^{1-x} x^p y^q\, (1 - x - y)^{r+1} dx\, dy$$

The area over which the integration to be carried is triangle AOB

i.e., $x \geq 0$, $y \geq 0$, $X + Y \leq 1$.

Let $x + y = u$, $y = uv$

i.e., $x = u(1 - v)$, $y = uv$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 - v & v \\ -u & u \end{vmatrix} = u$$

$\therefore dx\, dy = u\, du\, dv$

When x=0, $u(1 - v) = 0, i.e., u = 0\ or\ v = 1$

When y=0, $uv = 0$, i.e., $u = 0\ or\ v = 0$

When $x + y = 1$, $u = 1$

Hence the triangle AOB transforms into the area between

$u = 0, u = 1, v = 0$, $v = 1$ in the uv lane.

Hence the given integral is

$$= \frac{1}{r+1} \int_0^1 \int_0^1 u^p (1 - v)^p (uv)^q\, (1 - u)^{r+1} u\, du\, dv$$

$$= \frac{1}{r+1} \int_0^1 \int_0^1 u^{p+q+1} v^q \,(1-u)^{r+1} (1-v)^p \,du\,dv$$

$$= \frac{1}{r+1} \int_0^1 u^{p+q+1} \,(1-u)^{r+1} du \int_0^1 v^q \,(1-v)^p \,dv$$

$$= \frac{1}{r+1} \,\beta(p+q+2, r+2)\,\beta(q+1, p+1)$$

$$= \frac{1}{r+1} \frac{\Gamma(\mathrm{p}+\mathrm{q}+2)\,\Gamma(\mathrm{r}+2)\Gamma(\mathrm{q}+1)\Gamma(\mathrm{q}+1)}{\Gamma(\mathrm{p}+\mathrm{q}+\mathrm{r}+4)}$$

Alter: Put $x+y+z=u, y+z=uv, z=uvw$. The integral can be evaluated directly.

Example 4:

Prove that $\iiint \frac{dx\,dy\,dz}{(1-x^2-y^2-z^2)^{\frac{1}{2}}} = \frac{\pi^2}{8}$, the integration extended to all positive values of the variables for which the expression is real.

Solution:

Putting $x^2 = X, y^2 = Y, z^2 = Z$, the integral reduces to $\frac{1}{8} \iiint \frac{dx\,dy\,dz}{\sqrt{XYZ}} \,(1-X-Y-Z)^{-\frac{1}{2}}$

[where X,Y,Z are positive subject to X+Y+Z≤ 1]

$= \frac{1}{8} \int_0^1 X^{\frac{-1}{2}} \mathrm{dx} \int_0^{1-X} Y^{\frac{-1}{2}} \,dy \int_0^{1-X-Y} Z^{\frac{-1}{2}} \,(1-X-Y-Z)^{-\frac{1}{2}} dZ$

$= \frac{1}{8} \int_0^1 X^{\frac{-1}{2}} \mathrm{dx} \int_0^{1-X} Y^{\frac{-1}{2}} \,dy \int_0^{\frac{\pi}{2}} 2\,d\theta$

[by putting Z= $(1-X-Y-Z)sin^2\theta$]

$= \frac{\pi}{4} \int_0^1 X^{\frac{-1}{2}} (1-X)^{\frac{1}{2}} \mathrm{dx}$

$$= \frac{\pi}{4} \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$= \frac{\pi}{4} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\,(2)}$$

$= \frac{\pi}{4} \frac{1}{2} \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{\pi^2}{8}.$

UNIT-V

Geometric and Physical Applications of Integral calculus

Chapter 5: Sections 5.1- 5.4 & Chapter 6: Sections 6.1- 6.5

5.GEOMETRICAL APPLICATIONS OF INTEGRATION

5.1. Areas under plane curves: Cartesian co-ordinates.

We shall find a formula for the area bounded by the arc of the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the portion of the x-axis between them.

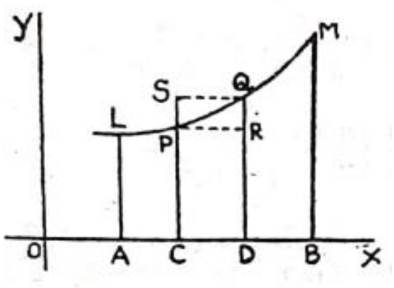

Figure 5.1

For definiteness, let us suppose that $a < b$. in the figure, the graph of $y = f(x)$ is LPQM.

Let $OA = a$ and $OB = b$. then the ordinates AL and BM are $y = f(a)$ and $y = f(b)$ respectively.

We shall prove that the area bounded by the arc LM, the ordinates AL and BM and the portions AB of the x-axis is $\int_a^b f(x)dx$.

Let P be any point (x, y) on the curve and Q a neighbouring point $(x + \Delta x, y + \Delta y)$ on it. Draw the ordinates PC and QD and draw PR and QS perpendicular to QD and CP respectively.

Let A represent the area bounded by the arc LP, the ordinates AL, CP and the portion AC of the x-axis. Then the area ALQD can be represented $A + \Delta A$ so that the area CPQD is ΔA. From the figure, it can be seen that the area CPQD is greater than the inner rectangle CPRD and is less than the outer rectangle CSQD.

Rectangle CPRD = CP. CD = $y\,\Delta x$ and

Rectangle CSQD = DQ. CD = $(y + \Delta y)\Delta x$.

$\therefore \Delta A > y\,\Delta x$ but $< (y + \Delta y)\Delta x$.

$\therefore y < \frac{\Delta A}{\Delta x} < (y + \Delta y).$

Proceeding to the limit when $\Delta x \to 0$.

$\frac{\Delta A}{\Delta x} \to \frac{dA}{dx}$ and $y + \Delta y \to y$.

$\therefore \frac{dA}{dx}$ lies between y and a quantity which tends to y in the imit.

Hence $\frac{dA}{dx} = y$.

$\therefore A = \int y dx + C$, where C is the constant of integration,

$$\text{i.e.}, A = \int f(x)dx + C.$$

Let us denote $\int f(x)dx$ by F(x), then A = F(x) + C.

When x = a, A = 0 as A is, by definition, area ALPC.

$\therefore 0 = F(a) + C \quad \ldots\ldots\ldots (1)$

When x = b, A = area ALMP which is sought.

$\therefore$ The required area ALMP = F(b) + C

= F(b) – F(a) on substituting for C from (1) $=[F(x)]_a^b = \int_a^b f(x)dx$.

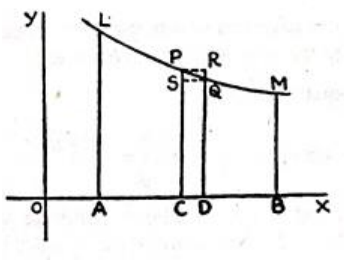

Figure 5.2

Note:

i) There is a point in the above proof which deserves our notice. In the curve we have drawn, y increases with x. If y decreases as x increases, as in the figure above, the same formula for the area holds good.

With the same notation, we find here that

$\Delta A < y\Delta x$ and $> (y + \Delta y)\Delta x$.

$\therefore y\Delta x > \Delta A > (y + \Delta y)\Delta x$.

$\therefore y > \frac{\Delta A}{\Delta x} > (y + \Delta y)$.

These inequalities are reversed in the case of an increasing function. But, in the limit, when $\Delta x \to 0$, $\frac{dA}{dx} = y$ and the rest of the proof is the same as before.

Whether y increases or decreases with x, the above formula holds good.

ii) One other point also deserves special mention. When part of the area is below the x-axis, the corresponding ordinates are negative and Δx being taken to be positive, the area will be negative.

iii) The formula for the area under a curve, the y-axis and the line $y = c. y = d$ is $\int_c^d x dy$ by a similar argument.

Example 1:

Find the area bounded by the curve $y^2 = 4ax$, the x-axis and the ordinate x = h.

Solution:

The curve is the parabola which, we know, passes through the origin. The limits for the area in question are 0 and h.

Hence the required area is

$$2\int_0^h \sqrt{4ax}dx = \frac{4\sqrt{a}}{3}\left(x^{\frac{3}{2}}\right)_0^h = \frac{4h\sqrt{ah}}{3}$$

(The area bounded by the above parabola and the double ordinate x = h is twice the above area by symmetry).

Example 2:

Find the area bounded by one arch of the curve $y = \sin ax$ and the x-axis.

Solution:

The curve crosses the x-axis when x = 0 and $\frac{n\pi}{a}$, where n is a positive or negative integer. The limits for one arch are 0 and $\frac{\pi}{a}$. Hence the area is

$\int_0^{\frac{\pi}{a}} \sin ax\, dx = -\frac{1}{a}(\cos ax)_0^{\frac{\pi}{a}} = -\frac{1}{a}\left\{\cos a\left(\frac{\pi}{a}\right) - \cos a(0)\right\} = -\frac{1}{a}(-1-1) = \frac{2}{a}$

Example 3:

Find the area bounded by one arch of the cycloid

$x = a(\theta - \sin\theta); y = a(1 - \cos\theta)$ and its base.

Solution:

As the point P describes one arch, the parameter θ varies from 0 to 2π.

$$\therefore \text{Area required} = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta$$

$$= \int_0^{2\pi} a(1-\cos\theta)a(1-\cos\theta)d\theta$$

$$= a^2\int_0^{2\pi}(1-\cos\theta)^2\, d\theta = a^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$$

$$= a^2\int_0^{2\pi}\left(1 + \frac{1+\cos 2\theta}{2}\right)d\theta \text{ as } \int_0^{2\pi}\cos\theta\, d\theta = [\sin\theta]\begin{matrix}2\pi\\0\end{matrix} = 0$$

$$= 3\pi a^2 \text{ as } \int_0^{2\pi}\cos 2\theta d\theta = \left[\frac{\sin 2\theta}{2}\right]\begin{matrix}2\pi\\0\end{matrix} = 0$$

Example 4:

Find the area of loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$.

Solution:

The limits for the loop are –a and 0.

As the curve is symmetrical about the x-axis,

the area of the loop = twice the area of the loop above the x-axis

$= 2\int_0^{-1} ydx = 2\int_0^{-a} x\left(\frac{a+x}{a-x}\right) dx.$

To integrate, put $x = a\cos 2\theta$

The integral reduces to $-2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} a^2 \cos 2\theta \frac{\cos\theta}{\sin\theta} \sin 2\theta d\theta$

$$= -4a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \cos^2\theta \, d\theta$$

$$= -2a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta (1 + \cos 2\theta) d\theta$$

$$= -a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos 2\theta + 1 + \cos 4\theta) d\theta$$

$$= -a^2 \left[\sin 2\theta + \theta + \frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2}$$

$$= -a^2 \left[\frac{\pi}{2} - 1 - \frac{\pi}{4}\right]$$

$$= a^2 \left(1 - \frac{\pi}{4}\right)$$

Hence the area required is $2a^2\left(1 - \frac{\pi}{4}\right)$.

5.2.Area of a closed curve:

Let AL and BM be the tangents to the closed curve parallel to the y-axis. Let an intermediate ordinate meet the curve in two points P_1 and P_2, where P_1 is (x, y_1) and P_2 is (x, y_2). Let $y_1 > y_2$. Denoting the OL and OM by a and b respectively, area $LAP_1BM = \int_a^b y_1 dx$ and area $LAP_2BM = \int_a^b y_2 dx$. By subtraction, we get the area of the closed curve to be $\int_a^b (y_1 - y_2) dx$.

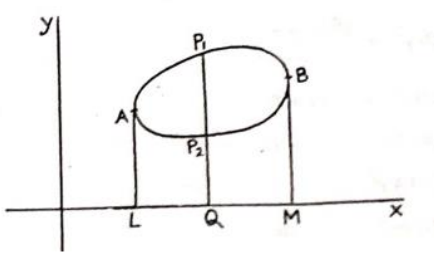

Figure 5.3

The integral gives the area whether the x-axis cuts the curve or not. The values y_1 and y_2 corresponding to any value of x are found by solving the equation of the curve as a quadratic in y.

Example 1:

Find the area of the ellipse $x^2 + 4y^2 - 6x + 8y + 9 = 0$

Solution:

Writing this as a quadratic in y,

$4y^2 + 8y + x^2 - 6x + 9 = 0$

If y_1 and y_2 be the roots,

$$y_1 + y_2 = -2 \; and \; y_1 y_2 = \frac{x^2 - 6x + 9}{4}$$

Hence $y_1 - y_2 = \sqrt{(y_1 + y_2)^2 - 4y_1 y_2}$

$= \sqrt{4 - (x^2 - 6x + 9)}$

$= \sqrt{6x - x^3 - 5}$

$= \sqrt{(1 - x)(x - 5)}$

The two values y_1 and y_2 are equal when x = 1 and x = 5.

These are the abscissae of the points at which the tangents are parallel to the y-axis.

Hence the area of the ellipse = $\int_1^5 (y_1 - y_2)dx$

$$= \int_1^5 \left(\sqrt{(1 - x)(x - 5)}\right)dx = 32 \int_0^{\pi/2} sin^2 \theta \, cos^2 \theta d\theta$$

(on putting $x = sin^2 \theta + 5 \, cos^2 \theta ; dx = -8 \, sin \, \theta \, cos \, \theta d\theta$)

$$= 32 \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = 2\pi$$

In some cases, two or more curves form the contour of an area. To find this, we must draw the curves and find the limits by solving for their points of intersection. The method is best exemplified by the following example.

Example 1:

Find the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4by$

Solution:

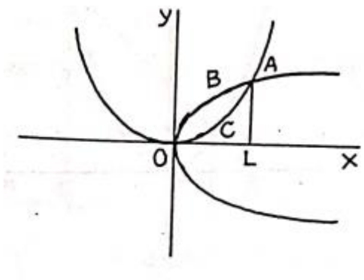

Figure 5.4

$y^2 = 4ax$ has Ox as its axis and $x^2 = 4by$ has Oy as its axis. To find the abscissae of their points of intersection, eliminate y between the equations. We get $x^4 = 16b^2.4ax$. Hence x = 0 or $4a^{\frac{1}{3}}b^{\frac{2}{3}}$; x = 0 corresponds to O the origin. $x = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$ corresponds to A the other point of intersection.

Area required = Area OBAL – Area OCAL.

Area OBAL = $\int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} ydx$ where $y^2 = 4ax$

$$= 2\sqrt{a}\int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \sqrt{x}dx$$

$$= \frac{4\sqrt{a}}{3}\left[x^{\frac{2}{3}}\right]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}}$$

$$= \frac{32}{3}ab$$

Area OCAL = $\int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} ydx$ where $x^2 = 4by$

$$= \frac{1}{4b}\int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} x^2dx = \frac{1}{12b}[x^3]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} = \frac{16}{3}ab$$

∴ The required area $\frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab$

Example 2:

Find the area enclosed between the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.

Solution:

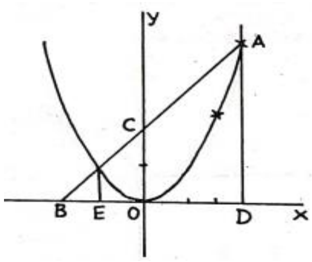

Figure 5.5

The straight line cuts off intercepts $-\frac{3}{2}$ and 3 from the axes and its graph is drawn. Let it cut the parabola at two points.

The abscissae of these points, viz., OD and OE are got by eliminating y between the two equations and solving the resulting quadratic. From the straight line equation, y = 2x + 3. Putting this in $y = x^2$, we have $x^2 = 2x + 3$, i.e., $x^2 - 2x - 3 = 0$.

$\therefore x = 3 \; or - 1$. Hence OD = 3 and OE = -1

Area required = $\int_{-1}^{3}(y_1 - y_2)dx$, where y_1 is the ordinate of the straight line corresponding to x, i.e., $y_1 = 2x + 3$ and y_2 the ordinate $y = x^2$,

i.e., $y_2 = x^2$

Hence the area = $\int_{-1}^{3}(2x + 3 - x^2)dx = \left(x^2 + 3x - \frac{x^3}{3}\right)_{-1}^{3}$

$$= 9 + 9 - 9 - \left(1 - 3 + \frac{1}{3}\right) = 10\frac{2}{3}$$

5.3. Areas in polar co-ordinates.

We propose to find a formula for the area bounded by the curve whose polar equation is

$r = f(\theta)$ and two radii vectors OA and OB. Let X$\hat{O}$A and X$\hat{O}$B be respectively α and β.

Let P be a point (r, θ) on the curve and P′ a neighboring point $(r + \Delta r, \theta + \Delta\theta)$ on it. If we denote the area AOP by A, then the area denoted by AOP′ is A + ΔA so that area POP′ is ΔA.

Let the circle centre O and radius OP and OP′ at M and the circle centre O and radius OP′ cut OP produced at N.

Area POP′ lies between the areas of the circular sectors OPM and OP′N,

i.e., $\frac{1}{2}r^2\Delta\theta$ and $\frac{1}{2}(r+\Delta r)^2\Delta\theta$.

$\therefore \frac{1}{2}(r+\Delta r)^2\Delta\theta > \Delta A > \frac{1}{2}r^2\Delta\theta$

Dividing by $\Delta\theta$ and proceesing to the limit as $\Delta\theta \to 0$, we have $\frac{dA}{d\theta} = \frac{1}{2}r^2$

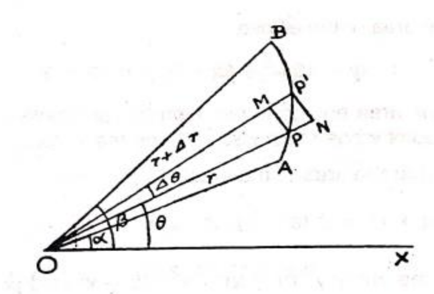

Figure 5.6

$$\therefore A = \frac{1}{2}\int r^2 d\theta + C = F(\theta) + C \; where \; F(\theta) = \frac{1}{2}\int r^2 d\theta$$

Putting $\theta = \alpha, A = 0.$ $\therefore 0 = F(\alpha) + C$ ……..(i)

Putting $\theta = \alpha, A = area\ OAB = F(\beta) + C$ ………. (ii)

∴By subtraction, area OAB = $F(\beta) - F(\alpha)$

$$= [F(\theta)]_{\alpha}^{\beta} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Example 1:

Find the area of the cardioid $r = a(1 + cos\,\theta)$.

Solution:

Since the curve is symmetrical about the initial line, the area

$$= 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$$

$$= a^2 \int_0^{\pi} (1 + \cos\theta)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} d\theta$$

$= 8a^2 \int_0^{\frac{\pi}{2}} \cos^4 \varphi \, d\varphi$ (on putting $\frac{\theta}{2} = \varphi$)

$$= 8a^2 \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{3\pi a^2}{2}$$

Example 2:

Find the entire area of the lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$

Solution:

The area consists of two loops and each loop is symmetrical about the initial line.

The area required = 4 x the area of one-half loop of the curve above the initial line.

$$= 4\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = 2a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta = a^2 (\sin 2\theta) \begin{matrix} \frac{\pi}{4} \\ 0 \end{matrix} = a^2$$

5.4.Approximate Integration:

We now give two rules for evaluating $\int_a^b f(x)\,dx$ approximately. These are useful when integration is impossible in terms of known functions.

5.4.1 Trapezoidal Rule:

The exact value of $\int_a^b f(x)dx$ gives the measure of the area bounded by the curve y = f(x), x = a, x = b and portion of the x-axis. We shall dissect this area into trapezoids and by adding their areas, evaluate the total area approximately.

Divide the segment b – a on Ox into n equal parts, each of length Δx. Let us denote the abscissae of the successive points of division by

$x_0(= a), x_1, x_2, \ldots., x_n(= b)$.

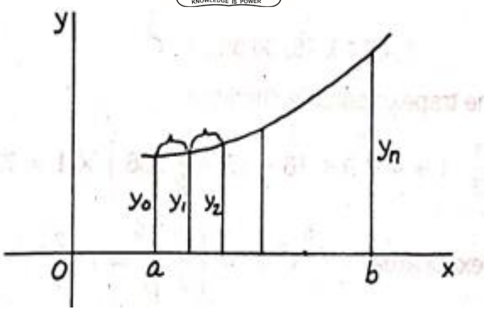

Figure 5.7

Let the corresponding ordinates be

$y_0 = f(x_0), y_1 = f(x_1), \ldots.., y_n = f(x_n)$

∴ By joining the extremities of consecutive ordinates, we form trapezoids. As the area of the trapezium is one-half the product of the sum of the parallel sides and the altitude, we get

Area of the first trapezium $= \frac{1}{2}(y_0 + y_1)\Delta x,$

Area of the second trapezium $= \frac{1}{2}(y_1 + y_2)\Delta x,$

……………..

Area of the last (n^{th}) trapezium $= \frac{1}{2}(y_{n-1} + y_n)\Delta x.$

Adding, the sum of the trapezoids $= \Delta x \left\{\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right\}$

This expression is an approximation to the required area.

Example 1:

Calculate $\int_1^6 x^2 dx$ by the trapezoidal rule.

Solution:

We shall, for illustration, divide the interval (1, 6) into five equal parts each of length 1.

$\therefore \Delta x = \frac{6-1}{5} = 1.$

When x = 1, 2, 3, 4, 5, 6 the corresponding ordinates y are 1, 4, 9, 16, 25, 36 as $y = x^2$.

By the trapezoidal rule, the area

$$= \left(\frac{1}{2}.1 + 4 + 9 + 16 + 25 + \frac{1}{2}.36\right).1 = 72.5$$

The exact area $= \int_1^6 x^2 dx = \left[\frac{x^3}{3}\right]_1^6 = 71\frac{2}{3}$.

Hence the error in the approximation is roughly 1 percent.

5.4.2. Simpson's Rule:

A closer approximation to the area than the trapezoidal rule is furnished by what is known as Simpson's rule. Here we join the extremities of successive ordinates by arcs of parabola and sum up the areas under these arcs.

Let the three points A, B, C on the given curve have ordinates y_1, y_2, y_3 whose abscissae are –h, o, h respectively. Let us take h to be small. If we pass a parabola through these points with its axis parallel to the y-axis, its equation is of the form $y = a + bx + cx^2$. The values of a, b, c are determined by expressing that $A(-h, y_1), B(0, y_2) and\ C(h, y_3)$ lie on the curve.

$\therefore y_1 = a - bh + ch^2; y_2 = a; y_3 = a + bh + ch^2$ … … … ..(1)

The area bounded by this parabola, the portion of the x-axis and the ordinates $x = \pm h$ is

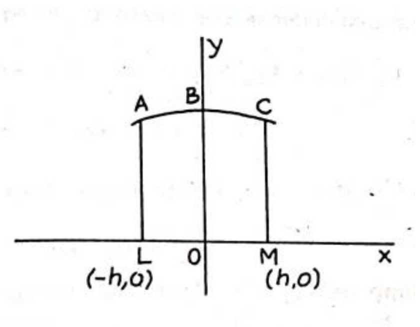

Figure 5.8

$$\int_{-h}^{+h} (a + bx + cx^2)dx = \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)_{-h}^{+h}$$

$$= 2\left(ah + \frac{ch^3}{3}\right) = \frac{2h}{3}(3a + ch^2)$$

From (1), $y_1 + y_3 = 2(a + ch^2)$

$\therefore y_1 + 4y_2 + y_3 = 6a + 2ch^2 = 2(3a + ch^2)$

Hence the area under the parabolic arc

$= \frac{h}{3}(y_1 + 4y_2 + y_3)$ ……….. (2)

This area is a close approximation to the area ALMC.

Let us take an odd number, i.e., 2n + 1 equidistant ordinates of the given curve and let the successive ordinates be denoted by $y_1, y_2, \ldots, y_{2n+1}$

The area under a parabolic arc passing through extremities of the ordinates $y_1, y_2\ and\ y_3 = \frac{h}{3}(y_1 + 4y_2 + y_3)$ by equation (2)

Similarly, the area under a parabolic arc passing through the extremities of the ordinates $y_3, y_4, y_5 = \frac{h}{3}(y_2 + 4y_4 + y_5)$ and so on.

The total area under several parabolic arcs so drawn,

$$= \frac{h}{3}\{(y_1 + 4y_2 + y_3) + (y_3 + 4y_4 + y_5) + \cdots + (y_{(2n-1)} + 4y_{2n} + y_{(2n+1)})\}$$

$$= \frac{h}{3}\{y_1 + y_{(2n+1)} + 2(y_3 + y_5 + \cdots + y_{(2n-1)}) + 4(y_2 + y_4 + \cdots + y_{2n})\}$$

Thus Simpson's rule is: To find an approximate value of the area under a given curve, divide it into an even number of strips by equidistant ordinates. Multiply one-third the distance between two consecutive ordinates by the sum of the first, the last, twice the sum of the other odd ordinates and four times the sum of all the even ordinates.

If the curve crosses the x-axis at one or both ends of the required area, one or both of the extreme ordinates must be taken as zero.

Example

Compute $\int_0^8 \frac{xdx}{1+x^2}$ by Simpson's rule.

Solution:

We shall divide the range 0 to 8 into eight equal intervals. Corresponding to the abscissae 0, 1, …., 8, the values of the ordinates are got from the equation $y = \frac{x}{1+x^2}$ and they are

$0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}$ *and* $\frac{8}{65}$

By Simpson's rule, the area

$$= \frac{1}{2}\left[0 + \frac{8}{65} + 2\left(\frac{2}{5} + \frac{4}{17} + \frac{6}{37}\right) + 4\left(\frac{1}{2} + \frac{3}{10} + \frac{5}{26} + \frac{7}{50}\right)\right]$$

$$= \frac{1}{2}[0.1231 + 1.5950 + 4.5392] = 2.0824$$

By actual integration, $\int_0^8 \frac{xdx}{1+x^2} = \frac{1}{2}[log_e(1+x^2)]_0^8$

$$= \frac{1}{2} log_e 65 = \frac{1}{2}\{log_{10} 65\, x log_e 10\} = 2.086$$

This error in using Simpson's rule here is 0.2 percent.

6.PHYSICAL APPLICATIONS OF INTEGRATION

6.1. Centroid:

Let a system of particles of masses $m_1, m_2, \ldots$ be situated at points whose co-ordinates are $(x_1, y_1), (x_2, y_2), \ldots$, with reference to fixed axes. The point whose co-ordinates are $(\bar{x}, \bar{y})$ given by the equations $\bar{x} = \frac{\sum mx}{\sum m}, \bar{y} = \frac{\sum my}{\sum m}$ is called the center of mass of the system.

If instead of a finite number of particles, we have a continuous distribution of matter, as in the case of a lamina or a rigid body, the summations in the above formulae become definite integrals. Thus the centre of mass of the body of mass M is given by the equations $M\bar{x} = \int xdm$ and $M\bar{y} = \int ydm$, where dm is an element of mass of the body concentrated at the point (x, y). If the lamina or the body be of uniform density, the centre of mass is known as the centroid.

6.2. Centre of mass of an arc:

Let P be any point (x, y) of a plane arc, whose actual distance from a fixed point on the curve is s, and ρ be the line density (i.e., mass per unit length of the curve) at P and let PQ be an elementary arc ds. Then P (x, y). the formulae for the centre of mass of the arc are $\bar{x} = \frac{\int \rho ds\, x}{\int \rho ds}$ and $\bar{x} = \frac{\int \rho ds\, y}{\int \rho ds}$ between suitable limits. ds is given by the formula $ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$ in

Cartesian co-ordinates and $ds = \left\{\left(\frac{dy}{d\theta}\right)^2 + r^2\right\}^{\frac{1}{2}} d\theta$ in polar co-ordinates. When the equation of the curve is given, $\frac{dy}{dx}$ *or* $\frac{dr}{d\theta}$ can be calculated as the case may be. If the arc be uniform, ρ is a constant and the centroid of the arc is given by $\overline{x} = \frac{\int xds}{\int ds}$ and $\overline{y} = \frac{\int yds}{\int ds}$.

Example 1:

Find the centroid of the arc of the parabola $y^2 = 4ax$ between the vertex and the point (x, y).

Solution:

Here $y\frac{dy}{dx} = 2a$.

$$\therefore ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \left(1 + \frac{4a^2}{y^2}\right)^{\frac{1}{2}} dx$$

$$= \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx$$

Hence $\overline{x} = \frac{\int_0^x xds}{\int_a^x ds} = \frac{\int_0^x \sqrt{x(x+a)}dx}{s}$ and $\overline{y} = \frac{\int_0^x yds}{\int_a^x ds} = 2\sqrt{a}\frac{\int_0^x \sqrt{(x+a)}dx}{s}$

(i) The denominator is

$$\frac{1}{4a}\left[y\sqrt{4a^2 + y^2} + 4a^2 \sin^{-1}\frac{y}{2a}\right]$$

$= \sqrt{x(x+a)} + a \log \frac{\sqrt{x}+\sqrt{x+a)}}{\sqrt{a}}$ on putting $y^2 = 4ax$.

(ii) $\int_0^x \sqrt{x(x+a)}dx = \int_0^x \left\{\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}\right\}^{\frac{1}{2}} da$

$= \frac{a^2}{4}\int_0^\theta \sinh^2\theta \, d\theta$ on putting $x + \frac{a}{2} = \frac{a}{2}\cosh\theta$;

$$= \frac{a^2}{8}\int_0^\theta (\cosh 2\theta - 1)d\theta = \frac{a^2}{8}\left(\frac{\sinh 2\theta}{2} - \theta\right)$$

$$= \frac{a^2}{8}(\sinh\theta \cosh\theta - \theta)$$

$$= \frac{1}{2}\left[\left(x + \frac{a}{2}\right)\left\{\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}\right\}^{\frac{1}{2}} - \frac{a^2}{4}\cosh^{-1}\frac{x + \frac{a}{2}}{\frac{a}{2}}\right]$$

$$= \frac{1}{2}\left(x + \frac{a}{2}\right)\sqrt{x^2 + ax} - \frac{a^2}{8} log\left[\frac{x + \frac{a}{2} + \sqrt{x^2 + ax}}{\frac{a}{2}}\right]$$

(iii) $\int_0^x \sqrt{x + a}\, dx = \frac{2}{3}[(x + a)^{\frac{3}{2}} - a^{\frac{3}{2}}]$

Substituting these values of the integrals, $\overline{x}$ *and* $\overline{y}$ are got.

6.3. Centre of mass of a plane area:

Let δA be an element of area surrounding or at the point (x, y) and ρ be the density at the point. The element of mass $dm = \rho dA$ and the formulae for the center of mass take the form

$\overline{x} = \frac{\int \rho dA\, x}{\int \rho dA}$ and $\overline{y} = \frac{\int \rho dA\, y}{\int \rho dA}$ .... (1)

If the area be of uniform density, let the ordinate at distance x cut the curve in points whose ordinates are y_1 *and* y_2. Now dA can be taken as $(y_1 - y_2)dx$, whose center of mass has co-ordinates x and $\frac{y_1+y_2}{2}$ in the limit when $\delta x \to 0$. Hence

$\overline{x} = \frac{\int x(y_1-y_2)dx}{\int(y_1-y_2)dx}$ *and* $\overline{y} = \frac{\int(y_1-y_2)\left(\frac{y_1+y_2}{2}\right)dx}{\int(y_1-y_2)dx}$ .... (2)

From the equation of the curve, the values of y_1 and y_2 are known and the limits for x are such as to cover the area in question.

If the area be bounded by the arc y = f(x), the ordinates x = a and x = b and the portion of the x-axis, then dA = ydx

And this acts $\left(x, \frac{y}{2}\right)$. Hence the centroid is given by

$\overline{x} = \frac{\int_a^b xy\, dx}{\int_a^b ydx}$ *and* $\overline{x} = \frac{\int_a^b xy\, dx}{\int_a^b ydx}$ .... (3)

If polar co-ordinates be employed and the area be uniform dA can be taken to be almost a triangle and equal to $\frac{1}{2}r^2 d\theta$. Its center of mass is the median point of this triangle practically, i.e., the point whose polar co-ordinates are $\left(\frac{2}{3}r, \theta\right)$ in the limit when $\delta\theta \to 0$.

Hence the centroid of the area is given by

$$\overline{x} = \frac{\int \frac{1}{2} r^2 d\theta \left(\frac{2}{3}\right) r \cos\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{2}{3} \frac{\int r^3 \cos\theta d\theta}{\int r^2 d\theta} \text{ and } \overline{y} = \frac{\int \frac{1}{2} r^2 d\theta \left(\frac{2}{3}\right) r \sin\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{2}{3} \frac{\int r^3 \sin\theta d\theta}{\int r^2 d\theta}$$

…………. (4)

Where the equation of the curve $r = f(\theta)$ is known and the limits for θ to cover the area required are determined in any problem.

Example 1:

Find the centroid of an elliptic quadrant.

Solution:

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Using the formulae (3) above,

$\overline{x} = \frac{\int_0^a xydx}{\int_0^a ydx}$ as x varies from 0 to a in the elliptic quadrant

$$= \frac{\int_0^a xb\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} dx}{\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} dx}$$

Putting $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$ in both the numerator and denominator.

$$\overline{x} = a\frac{\int_0^{\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta}{\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta} = \frac{a\left[\frac{1}{3}\right]}{\frac{\pi}{4}} = \frac{4a}{3\pi}$$

By symmetry, $\overline{y} = \frac{4b}{3\pi}$

Corollary:

The centroid of a circular quadrant is $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$

Example 2:

Find the centroid of the arc and sector of a circle.

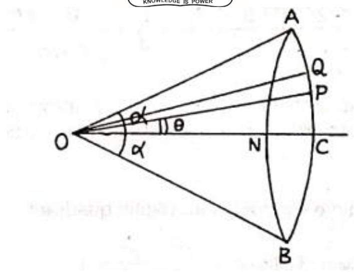

Figure 6.1

Solution:

Let the arc AB of a circle subtend an angle 2α radians at its center O and let OC be the radius bisecting the arc. By symmetry, the centre of the arc lies on OC.

Taking OC as the x-axis, let P be any point (x, y) on the arc and let COP be θ and arc CP be s.

Then $x = a\cos\theta \ \ and \ s = a\theta.$

Hence, for the arc,

$$\overline{x} = \frac{\int xds}{ds} = \frac{\int_{-a}^{+a} a\cos\theta \, ad\theta}{\int_{-a}^{+a} ad\theta} = a\frac{\sin\alpha}{\alpha}$$

For the sector AOB, the centroid lies on OC by symmetry. The element of area is $\frac{1}{2}r^2 d\theta$

$$\overline{x} = \int_{-a}^{+a} \frac{1}{2}a^2 \, d\theta \left(\frac{2}{3}\right) a\cos\theta = \frac{2}{3}a\frac{\sin\alpha}{\alpha}$$

Corollary: Putting $\alpha = \frac{\pi}{2}$, the centroids of a semi-circular arc and area lie on the middle radius at distances $\frac{2a}{\pi}$ and $\frac{4a}{3\pi}$ from the center.

Example 3:

Find the centroid of the arc enclosed by the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ and the x-axis from cusp to cusp.

Solution:

The cusps correspond to $\theta = 0$ and $\theta = 2\pi$.

As the cycloid is symmetrical about the line $x = \pi$ the centroid lies on the line of symmetry.

Hence $\overline{x} = a\pi$ and $\overline{y}$ need only be evaluated.

From (3) of 1.3.

$$\overline{y} = \frac{\int \frac{y^2}{2} dx}{\int y dx} = \frac{a}{2} \cdot \frac{\int_0^{2\pi} (1 - \cos\theta)^3 d\theta}{\int_0^{2\pi} (1 - \cos\theta)^2 d\theta}$$

$$= \frac{a}{2} \cdot \frac{\int_0^{2\pi} 8 \cos^6 \frac{\theta}{2} d\theta}{\int_0^{2\pi} 4 \cos^4 \frac{\theta}{2} d\theta} = a \frac{\int_0^{\pi} \cos^6 \phi \, d\phi}{\int_0^{\pi} \cos^4 \phi \, d\phi}$$

On putting $\frac{\theta}{2} = \phi$

$$= a \frac{2 \int_0^{\frac{\pi}{2}} \cos^6 \phi \, d\phi}{2 \int_0^{\frac{\pi}{2}} \cos^4 \phi \, d\phi} = a \frac{\frac{5}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}}{\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}} = \frac{5a}{6}$$

Example 4:

Find the centroid of the one loop of the lemniscate $r^2 = a^2 \cos 2\theta$

Solution:

For the tracing of the curve, vide Vol .1.

By symmetry, the centroid lies on the initial line

Hence $\overline{y} = 0$ and $\overline{x}$ need only be calculated.

From (4) of 1.3.,

$$\overline{x} = \frac{\int_0^{\frac{\pi}{4}} r^2 \cos\theta \, d\theta}{\int_0^{\frac{\pi}{4}} r^2 d\theta}$$

as θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ in a loop;

$$= \frac{2a}{3} \frac{2 \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta \, d\theta}{2 \int_0^{\frac{\pi}{4}} (\cos 2\theta) \, d\theta} = \frac{2}{3} a \frac{\int_0^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{\frac{3}{2}} \cos\theta \, d\theta}{\left\{\frac{\sin 2\theta}{2}\right\}_0^{\frac{\pi}{4}}}$$

$$= \frac{2\sqrt{2}a}{3} \int_0^{\frac{\pi}{2}} cos^4 \phi \, d\phi \;\; on \; putting \; \sqrt{2} \sin\theta = \sin\phi$$

in the numerator;

$$= \frac{2\sqrt{2}a}{3} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi a\sqrt{2}}{8}$$

6.4. Centroid of a solid of revolution:

Let the arc LM of the curve $y = f(x)$ revolve about x-axis and generated a solid of revolution. The element of area RSQP between the ordinates at distances x and $x + \Delta x$ from O generates the element of volume which is almost in the form of a cylinder. Hence the element of volume is $\pi y^2 \Delta x$ and its centroid lies on Ox at a distance x from O, when $\Delta x \to 0$. Taking the solid to be of uniform density, dm is proportional to $\pi y^2 \delta x$.

$$\overline{x} = \frac{\int_a^b x\pi y^2 dx}{\int_a^b \pi y^2 dx} = \frac{\int_a^b y^2 x dx}{\int_a^b y^2 dx} \; and \; \overline{y} = 0$$

Example:

Find the centroid of a uniform solid hemisphere.

Solution:

A hemisphere is generated by revolving the quadrant of the circle $x^2 + y^2 = a^2$ about one of its bounding radii taken to be the x-axis.

$$\overline{x} = \frac{\int_0^a y^2 x dx}{\int_a^b y^2 dx} = \frac{\int_0^a (a^2 - x^2) dx}{\int_0^a (a^2 - x^2) dx} \; as \; y^2 = a^2 - x^2$$

$$= \frac{\left(a^2\left(\frac{x^2}{2}\right) - \frac{x^4}{4}\right)_0^a}{\left(a^2 x - \frac{x^3}{3}\right)_0^a} = \frac{3a}{8} \; obviously \; \overline{y} = 0$$

6.5. Centroid of surface of revolution:

By revolving arc LM (fig 10) about the x-axis, we get a surface of revolution. The elementary area dS generated by the revolution of the arc RS about the x-axis is $2\pi y ds$ and this almost a frustum of a cone. Its centroid lies on the x-axis at distance x from the origin, when $\Delta x \to 0$. Taking the surface to be of uniform density, dm is proportional to $2\pi y ds$.

Hence, $\overline{x} = \frac{\int 2\pi yds.x}{\int 2\pi yds} = \frac{\int yxds}{\int yds}$ $and\ \overline{y} = 0$

The limits are to be fixed suitably in each case by drawing the figure of the generator curve.

$$ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx\ or\ ds = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$$

is used according as Cartesian or polar co-ordinates are employed.

Example 1:

Find the centroid of a hollow hemisphere.

Solution:

A hollow hemisphere is generated by revolving the arc AB of a quadrant of a circle about Ox. (fig 11 of ex.1)

Arc AP = s =$a\theta$ and $y = asin\,\theta\,; x = acos\,\theta$ and from A to B, θ varies from 0 to $\frac{\pi}{2}$.

$$\overline{x} = \frac{\int yds.x}{\int yds} = a.\frac{\int_0^{\frac{\pi}{2}} cos\,\theta\,sin\,\theta\,d\theta}{\int_0^{\frac{\pi}{2}} sin\,\theta\,d\theta}$$

$$= a\frac{\left[\frac{sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}}{[-cos\,\theta]_0^{\frac{\pi}{2}}} = \frac{a}{2}$$

Evidently, $\overline{y} = 0$

Example 2:

Find the centroid of the surface generated by revolving the cardioid $r = a(1 + cos\,\theta)$ about the initial line.

Solution:

$ds = 2\,acos\left(\frac{\theta}{2}\right)d\theta$ (ex.4 of 5)

$$\overline{x} = \frac{\int yds.x}{\int yds} = \frac{\int_0^{\pi} r^2\,sin\,\theta\,cos\,\theta\,cos\frac{\theta}{2}d\theta}{\int_0^{\pi} r\,sin\,\theta\,cos\frac{\theta}{2}d\theta}$$

$as\ x = r\cos\theta\ and\ y = r\sin\theta$

$$= a\frac{\int_0^{\pi}(1+\cos\theta)^2\sin\theta\cos\theta\cos\frac{\theta}{2}d\theta}{\int_0^{\pi}(1+\cos\theta)\sin\theta\cos\frac{\theta}{2}d\theta}\ on\ substituting\ r = a(1+\cos\theta)$$

$$= 2a\frac{\int_0^{\pi}\cos^6\frac{\theta}{2}\sin\frac{\theta}{2}(2\cos^2\frac{\theta}{2}-1)\,d\theta}{\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta}$$

$$\text{as } \cos\theta = 2\cos^2\frac{\theta}{2} - 1 \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2a\frac{\left[-\frac{4}{9}\cos^9\frac{\theta}{2} + \frac{2\sin^7\frac{\theta}{2}}{7}\right]_0^{\pi}}{\left[-\frac{2}{5}\cos^5\frac{\theta}{2}\right]_0^{\pi}}$$

$$= 2a\frac{(\frac{4}{9}-\frac{2}{7})}{\frac{2}{5}} = \frac{50a}{63}$$

Example 3:

Find the centroid of a hemispherical distribution of mass in which the density varies as the nth power of the distance from the centre.

Solution:

Dissect the solid hemisphere into a number of thin concentric shells as in the figure. Let the radius of a typical shell ABC be x and its thickness dx. The density ρ of the material pf this shell is kx^n, where k is a constant.

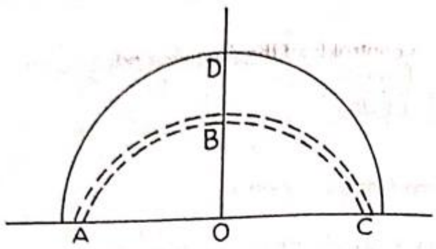

Figure 6.2

Hence, the element of mass $dm = 2\pi x^2 dxkx^n$ and its center of mass is at a distance $\frac{x}{2}$ from O on OD, the normal at O to the plane base, as this shell may be regarded as a hollow hemisphere of radius x. (Example.1)

∴ The centre of mass of the aggregated of the shells, i.e., the hemisphere lies on OD and its distance $\bar{x}$ from O is given by

$$\bar{x} = \frac{\int_0^a 2\pi x^2 dxkx^n \left(\frac{x}{2}\right)}{\int_0^a 2\pi x^2 dxkx^n} = \frac{1}{2}\frac{\int_0^a x^{n+3}dx}{\int_0^a x^{n+2}dx} = \frac{a}{2}.\frac{n+3}{n+4}$$

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